Commun. Comput. Phys. Vol. 27, No. 1, pp. 227-250
doi: 10.4208/cicp.OA-2018-0220 January 2020

The Riemann Problem for a Blood Flow Model in Arteries

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Received 31 August 2018; Accepted (in revised version) 31 January 2019

Abstract. In this paper, the Riemann solutions of a reduced 6×6 blood flow model in medium-sized to large vessels are constructed. The model is nonstrictly hyperbolic and non-conservative in nature, which brings two difficulties of the Riemann problem. One is the appearance of resonance while the other one is loss of uniqueness. The elementary waves include shock wave, rarefaction wave, contact discontinuity and stationary wave. The stationary wave is obtained by solving a steady equation. We construct the Riemann solutions especially when the steady equation has no solution for supersonic initial data. We also verify that the global entropy condition proposed by C.Dafermos can be used here to select the physical relevant solution. The Riemann solutions may contribute to the design of numerical schemes, which can apply to the complex blood flows.

AMS subject classifications: 35L65, 35L04, 58J45, 76N10, 35L60

Key words: Blood flow, elementary waves, Riemann problem, non-uniqueness, global entropy condition.

1 Introduction

A simple set of equations for the blood flow in medium-length to large arteries and veins are given by [28]

$$\begin{cases} A_t+(Au)_x=0, \\ (Au)_t+(Au^2)_x+\dfrac{A}{\rho}p_x=-Ru, \end{cases} \tag{1.1}$$

where $A(x,t)$ is the cross section area of the vessel, ρ,p,u represent the density, the pressure and the averaged velocity of the blood, respectively. We treat ρ as a constant. R is the

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flow resistance per unit length of the tube, assumed to be a known function here. See [27] for more details.

To complete the system, an additional condition on the pressure p is provided by the tube law, which is analogous to the state equation of fluid flows. Following [23], we have the tube law

$$p = p_e(x,t) + \psi(A;A_E,K), \tag{1.2}$$

where

$$\psi(A;A_E,K) = K(x)\big(\alpha^m - \alpha^n\big), \quad \alpha = \frac{A}{A_E}. \tag{1.3}$$

The p_e is the external pressure of the vessel. K is the stiffness coefficient of the vessel, which represents the elastic properties of the vessel. A_E is the cross section area at equilibrium state. $m \geq 0$ and $n \leq 0$ are two constants. For flows in arteries, $m = 1/2$, $n = 0$, see [22]. In this paper, we take $0 < m < 1$, $n = 0$ for simplicity.

In [28], Toro and Siviglia took p_e, K and A_E as a function of x only. Moreover, they added the following conditions to complete system (1.1)

$$\partial_t p_e = 0, \quad \partial_t K = 0, \quad \partial_t A_E = 0. \tag{1.4}$$

Substituting (1.3) into the second equation of (1.1), we have

$$(Au)_t + (Au^2)_x + \frac{A}{\rho}\psi_A A_x + \frac{A}{\rho}\psi_K K_x + \frac{A}{\rho}\psi_{A_E}(A_E)_x + \frac{A}{\rho}(p_e)_x = -Ru. \tag{1.5}$$

Following Toro and Siviglia [28], we add an advection equation for a passive tracer ϕ representing the concentration of a chemical species. The tracer is transported passively with the fluid speed, so we have

$$\partial_t(A\phi) + \partial_x(Au\phi) = 0. \tag{1.6}$$

The advective equation is decoupled from the other equations. We note (1.6) does not add new difficulties to the Riemann problem. But for future applications, it is convenient to consider all six equations as follows

$$\partial_t U + Q(U)\partial_x U = S(U), \tag{1.7}$$

where $U = \big(u,A,K,A_E,p_e,\phi\big)$, and

$$Q(U) = \begin{pmatrix} u & \frac{1}{\rho}\psi_A & \frac{1}{\rho}\psi_K & \frac{1}{\rho}\psi_{A_E} & \frac{1}{\rho} & 0 \\ A & u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u \end{pmatrix}, \quad S(U) = \begin{pmatrix} -\frac{Ru}{A} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{1.8}$$