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High Order Arbitrary Lagrangian-Eulerian Finite Difference WENO Scheme for Hamilton-Jacobi Equations[†]

Yue Li[1], Juan Cheng[2,3,*], Yinhua Xia[4] and Chi-Wang Shu[5]

[1] *Graduate School, China Academy of Engineering Physics, Beijing 100088, P.R. China.*

[2] *Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, P.R. China.*

[3] *Center for Applied Physics and Technology, Peking University, Beijing 100871, P.R. China.*

[4] *School of Mathematics Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.*

[5] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*

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Abstract. In this paper, a high order arbitrary Lagrangian-Eulerian (ALE) finite difference weighted essentially non-oscillatory (WENO) method for Hamilton-Jacobi equations is developed. This method is based on moving quadrilateral meshes, which are often used in Lagrangian type methods. The algorithm is formed in two parts: spatial discretization and temporal discretization. In the spatial discretization, we choose a new type of multi-resolution WENO schemes on a nonuniform moving mesh. In the temporal discretization, we use a strong stability preserving (SSP) Runge-Kutta method on a moving mesh for which each grid point moves independently, with guaranteed high order accuracy under very mild smoothness requirement (Lipschitz continuity) for the mesh movements. Extensive numerical tests in one and two dimensions are given to demonstrate the flexibility and efficiency of our moving mesh scheme in solving both smooth problems and problems with corner singularities.

AMS subject classifications: 65M06, 35F21

Key words: ALE method, finite difference method, WENO method, Hamilton-Jacobi equation.

[†]This paper is dedicated to the honor of Professor Jie Shen on the occasion of his 60th Birthday.

*Corresponding author. *Email addresses:* `liyue9443@outlook.com` (Y. Li), `cheng_juan@iapcm.ac.cn` (J. Cheng), `yhxia@ustc.edu.cn` (Y. Xia), `chi-wang_shu@brown.edu` (C.-W. Shu)

1 Introduction

In this paper, we deal with the development of high order accurate weighted essentially non-oscillatory (WENO) finite difference schemes for solving one and two dimensional time-dependent Hamilton-Jacobi (HJ) equations on a moving mesh,

$$\phi_t + H(\nabla\phi, \mathbf{x}, t) = 0, \quad \text{in } \Omega\times(0,T), \quad \mathbf{x}\in\Omega, \tag{1.1}$$

where H is a nonlinear function which is at least Lipschitz continuous with respect to $\nabla\phi$. H may also depend on ϕ in applications, however the main difficulty for numerical solutions is the possibly nonlinear dependency of H on $\nabla\phi$.

The viscosity solution of (1.1) is introduced by Crandall and Lions [7], in which it is proved that there exists a unique bounded and Lipschitz continuous viscosity solution for the problem (1.1). We should note that the derivatives of the viscosity solution can be discontinuous (i.e. the development of corner-type singularities) even when the initial condition is smooth.

Hamilton-Jacobi equations are widely used in many areas, including computer vision and image processing, and front propagation problems in models for flame propagation and the growth of crystal [20, 24]. In the numerical simulations of multidimensional fluid flow, there are two typical choices: the Lagrangian framework, in which the mesh moves with the local fluid velocity, and the Eulerian framework, in which the fluid flows through a grid fixed in space. Hirt et al. proposed an arbitrary Lagrangian Eulerian (ALE) method [11]. Its basic idea is that the computational grid is no longer fixed or attached to fluid particles, but can move arbitrarily with respect to the coordinate system. In particular, it is often desirable that the mesh moves with the fluid at the interface (or at least in the normal direction of the interface or free surface), which ensures high resolution at the interface, while the movement of the mesh elsewhere could have more freedom. An arbitrary Lagrangian-Eulerian approach is very advantageous to solve multimaterial and moving boundary problems with incompressible flows [5] and compressible flows [19].

The monotone first-order accurate numerical method for solving Hamilton-Jacobi equations was first presented by Crandall and Lions [8]. Later, Osher and Sethian used the connection between conservation laws and Hamilton-Jacobi equations to construct high-order accurate "artifactfree" numerical methods [21]. After Osher and Shu [22] proposed a general framework for the numerical solution of Hamilton-Jacobi equations using successful methods from hyperbolic conservation laws, many high order numerical methods such as the essentially non-oscillatory (ENO) method [22], the weighted essentially non-oscillatory (WENO) method [13] and the discontinuous Galerkin (DG) method [12, 17] have been proposed along this route.

ENO and WENO schemes are high order accurate finite difference or finite volume schemes designed for problems with piecewise smooth solutions containing discontinuities. The key point of these methods is the adoption of an adaptive stencil for high order interpolation or reconstruction so as to avoid shocks, high or discontinuous gradient