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Efficient Numerical Solution of Dynamical Ginzburg-Landau Equations under the Lorentz Gauge

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Abstract. In this paper, a new numerical scheme for the time dependent Ginzburg-Landau (GL) equations under the Lorentz gauge is proposed. We first rewrite the original GL equations into a new mixed formulation, which consists of three parabolic equations for the order parameter ψ, the magnetic field $\sigma = \mathrm{curl}\mathbf{A}$, the electric potential $\theta = \mathrm{div}\mathbf{A}$ and a vector ordinary differential equation for the magnetic potential $\mathbf{A}$, respectively. Then, an efficient fully linearized backward Euler finite element method (FEM) is proposed for the mixed GL system, where conventional Lagrange element method is used in spatial discretization. The new approach offers many advantages on both accuracy and efficiency over existing methods for the GL equations under the Lorentz gauge. Three physical variables ψ, σ and θ can be solved accurately and directly. More importantly, the new approach is well suitable for non-convex superconductors. We present a set of numerical examples to confirm these advantages.

AMS subject classifications: 65M12, 65M22, 65M60

Key words: Ginzburg-Landau equations, Lorentz gauge, fully linearized scheme, FEMs, magnetic field, electric potential, superconductivity.

1 Introduction

This paper is concerned with efficient numerical methods for the time-dependent Ginzburg-Landau (GL) equations

$$\begin{cases} \eta \dfrac{\partial \psi}{\partial t} + i\eta\kappa\Phi\psi + \left(\dfrac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi + (|\psi|^2 - 1)\psi = 0, & \text{in } \Omega \times (0,T], \quad (1.1) \\ \dfrac{\partial \mathbf{A}}{\partial t} + \nabla\Phi + \mathbf{curl}\,\mathbf{curl}\,\mathbf{A} + \mathrm{Re}\left(\dfrac{i}{\kappa}\psi^* \nabla\psi\right) + |\psi|^2 \mathbf{A} = \mathbf{curl}\mathbf{H}_e, & \text{in } \Omega \times (0,T], \quad (1.2) \end{cases}$$

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with the following boundary and initial conditions

$$\left(\frac{i}{\kappa}\nabla\psi+\mathbf{A}\psi\right)\cdot\mathbf{n}=0, \quad \mathbf{curl}\,\mathbf{A}\times\mathbf{n}=\mathbf{H}_e\times\mathbf{n}, \qquad \text{on } \partial\Omega\times[0,T], \tag{1.3}$$

$$\psi(x,0)=\psi_0(x), \quad \mathbf{A}(x,0)=\mathbf{A}_0(x), \qquad \text{in } \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^3$. In the GL equations (1.1)-(1.4), the complex scalar function ψ is the order parameter, the real vector-valued function $\mathbf{A}$ is the magnetic potential, and the real scalar function Φ is the electric potential. ψ^* denotes the complex conjugate of the function ψ. Physically, $|\psi|^2$ denotes the density of the superconducting electron pairs. $|\psi|^2=1$ and $|\psi|^2=0$ represent the perfectly superconducting state and the normal state, respectively, while $0<|\psi|^2<1$ represents a mixed (vortex) state. The real vector-valued function $\mathbf{H}_e$ is the external magnetic field, κ (positive) is the Ginzburg-Landau parameter and η (positive) is a dimensionless constant. In the rest of this paper, we set $\eta=1$ for the sake of simplicity.

We refer to [3,11] for the detailed description of the Ginzburg-Landau model in superconductivity. Theoretical analyses of the GL equations have been well done, see [3,8,20] and references therein. Numerical methods for solving the GL equations have also been investigated extensively; see [1,6,7,10,13–21,23–29]. It is well-known that the GL equations admit the gauge invariance property, see [11,12]. Two popular gauges are the temporal gauge and the Lorentz gauge. Under the temporal gauge, the GL equations are defined by

$$\begin{cases} \dfrac{\partial\psi}{\partial t}+\left(\dfrac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi=0, & \text{in } \Omega\times(0,T], \quad (1.5)\\[2mm] \dfrac{\partial\mathbf{A}}{\partial t}+\mathbf{curl}\,\mathbf{curl}\,\mathbf{A}+\mathrm{Re}\left(\dfrac{i}{\kappa}\psi^*\nabla\psi\right)+|\psi|^2\mathbf{A}=\mathbf{curl}\,\mathbf{H}_e, & \text{in } \Omega\times(0,T]. \quad (1.6)\end{cases}$$

As fewer terms are involved, the GL equations under the temporal gauge looks simpler. We refer to [10,16,18,24,25,27,28] for the numerical methods for the GL equations under the temporal gauge. However, it should be noted that Eq. (1.6) for $\mathbf{A}$ is a degenerate parabolic equation, where $\|\mathbf{curl}\,\mathbf{A}\|_{L^2}$ is not equivalent to $|\mathbf{A}|_{H^1}$. Due to this degeneracy, in [10,23] an extra perturbation term $-\epsilon\nabla\mathrm{div}\mathbf{A}$ was added to Eq. (1.6) for $\mathbf{A}$. Therefore, the results obtained in [10,24] depend on the parameter ϵ. By taking $\Phi=-\mathrm{div}\mathbf{A}$, the GL equations under the Lorentz gauge can be written as

$$\begin{cases} \dfrac{\partial\psi}{\partial t}-i\kappa(\mathrm{div}\mathbf{A})\psi+\left(\dfrac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+(|\psi|^2-1)\psi=0, & \text{in } \Omega\times(0,T], \quad (1.7)\\[2mm] \dfrac{\partial\mathbf{A}}{\partial t}-\nabla\mathrm{div}\mathbf{A}+\mathbf{curl}\,\mathbf{curl}\,\mathbf{A}+\mathrm{Re}\left(\dfrac{i}{\kappa}\psi^*\nabla\psi\right)+|\psi|^2\mathbf{A}=\mathbf{curl}\,\mathbf{H}_e, & \text{in } \Omega\times(0,T], \quad (1.8)\end{cases}$$