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A Matrix Representation of Outer Derivations from $\mathfrak{gl}_{0|2}$ to the Generalized Witt Lie Superalgebra

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Abstract: Let $\mathfrak{gl}_{0|2}$ be a subalgebra of the general linear Lie superalgebra. In this paper, outer derivations from $\mathfrak{gl}_{0|2}$ to the generalized Witt Lie superalgebra are completely determined by matrices.

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1 Introduction

The non-inner derivations are called outer derivations, which are in fact equal to the first cohomology groups of (super)algebras under consideration. For Lie (super)algebras, the existence of outer derivations have been well investigated (see [1]–[5] for examples). During the past half century, the theory of outer derivations and first cohomology groups for modular Lie (super)algebras has developed in a variety of directions and a large number of results have been obtained (see [6]–[8] for example). For classical Lie superalgebras over a field of prime characteristic, the recent papers [9] and [10] computed low-dimensional cohomology groups of the special linear Lie superalgebra $\mathfrak{sl}_{m|n}$ and its subalgebra $A(1;0)$ with coefficients in Witt or special superalgebras by virtue of the direct sum decomposition of submodules and the weight space decompositions of these submodules relative to their standard Cartan subalgebra.

The original motivation for this paper comes from [11]. The treatment of linear superal-

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gebras necessitates matrix computational techniques which are set forth in [12]. This paper contains a considerable amount of computation and determines the outer derivations from $\mathfrak{gl}_{0|2}$ to the generalized Witt Lie superalgebra W. Section 2 reviews the necessary notions. Section 3 calculates derivations and inner derivations from $\mathfrak{gl}_{0|2}$ into each irreducible submodule of W over a field of prime characteristic. Therefore, outer derivations from $\mathfrak{gl}_{0|2}$ to W are determined. In Section 4, the outer derivations from $\mathfrak{gl}_{0|2}$ to W over a field of characteristic zero are considered. Throughout this paper $\mathbb{F}$ is assumed to be an arbitrary field. The set of positive integers and the set of nonnegative integers are written as $\mathbf{N}_+$ and $\mathbf{N}$, respectively. Let L be a Lie algebra over $\mathbb{F}$ and A be an arbitrary L-module. Let $x \in L$, $a \in A$, we denote by $x \cdot a$ the element x acts on a.

2 Preliminaries

An $\mathbb{F}$-linear mapping $\varphi\colon L \to A$ is called a derivation from L into A if

$$\varphi([x, y]) = x \cdot \varphi(y) - y \cdot \varphi(x), \qquad x, y \in L. \tag{2.1}$$

Denote by $\mathrm{Der}(L, A)$ the derivation space from L into A. A derivation $\psi_a\colon L \to A$ is called inner if there is $a \in A$ such that

$$\psi_a(x) = x \cdot a, \qquad x \in L.$$

Denote by $\mathrm{Ider}(L, A)$ the inner derivation space from L into A. Denote by $\mathrm{Oder}(L, A)$ the outer derivation space from L into A. Thus

$$\mathrm{Oder}(L, A) = \mathrm{Der}(L, A)/\mathrm{Ider}(L, A).$$

This implies that the first cohomology group $\mathrm{H}^1(L, A)$ is isomorphic to $\mathrm{Oder}(L, A)$.

The following lemma is a standard fact. For more details see [13].

Lemma 2.1 *Suppose that A is an L-module and $A_1, A_2, \cdots, A_k$ are submodules of A such that $A = A_1 \bigoplus A_2 \bigoplus \cdots \bigoplus A_k$. Then*

$$\mathrm{H}^n(L, A) = \bigoplus_{i=1}^{k} \mathrm{H}^n(L, A_i), \qquad n \in \mathbf{N}.$$

According to Lemma 2.1, we obtain

$$\mathrm{Oder}(L, A) = \bigoplus_{i=1}^{k} \mathrm{Oder}(L, A_i), \tag{2.2}$$

which is frequently used in the sequel.

For sake of simplicity, let m, n denote fixed integers in $\mathbf{N}_+ \setminus \{1, 2\}$. For $\alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m$, we put

$$|\alpha| := \sum_{i=1}^{m} \alpha_i.$$

Let $\mathcal{O}(m, \underline{t})$ denote the divided power algebra over $\mathbb{F}$ with an $\mathbb{F}$-basis $\{x^{(\alpha)} \mid \alpha \in \mathcal{O}(m, \underline{t})\}$, where

$$\mathcal{O}(m, \underline{t}) := \left\{\alpha := (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m \mid 0 \le \alpha_i \leqslant p^{t_i} - 1, \ i = 1, 2, \cdots, m\right\}.$$

Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ in n variables $\xi_1, \xi_2, \cdots, \xi_n$ and $\mathcal{O}(m, n, \underline{t})$ denote the tenser product $\mathcal{O}(m, \underline{t}) \otimes_{\mathbb{F}} \Lambda(n)$.

For $g \in \mathcal{O}(m, \underline{t})$, $f \in \Lambda(n)$, we write gf for $g \otimes f$. The following formulas hold in $\mathcal{O}(m, n, \underline{t})$:

$$x^{(\alpha)}x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}, \qquad \alpha, \beta \in \mathbf{N}^m;$$
$$\xi_i\xi_j = -\xi_j\xi_i, \qquad i, j = 1, 2, \cdots, n;$$
$$x^{(\alpha)}\xi_j = \xi_j x^{(\alpha)}, \qquad \alpha \in \mathbf{N}^m,\ j = 1, 2, \cdots, n,$$

where

$$\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^{m} \binom{\alpha_i+\beta_i}{\alpha_i}.$$

Put $Y_0 := \{1, 2, \cdots, m\}$ and $Y_1 := \{1, 2, \cdots, n\}$. Set

$$\mathbb{B}_k := \{\langle i_1, i_2, \cdots, i_k\rangle \mid 1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n\}$$

and

$$\mathbb{B} := \bigcup_{k=0}^{n} \mathbb{B}_k,$$

where $\mathbb{B}_0 := \emptyset$. For $u = \langle i_1, i_2, \cdots, i_k\rangle \in \mathbb{B}_k$, set $|u| := k$, $|\emptyset| := 0$, $\xi^{\emptyset} := 1$, $\xi^u := \xi_{i_1}\xi_{i_2}\cdots\xi_{i_k}$ and $\xi^E := \xi_1\xi_2\cdots\xi_n$. If $u \in \mathbb{B}_k$, $j \in \{u\}$, then $\{u - \langle j\rangle\} := \{u\} \setminus \{j\} \in \mathbb{B}_{k-1}$. Let $u(j) = |\{l \in \{u\} \mid l < j\}|$. If $j \in Y_1 \setminus \{u\}$, then we put $u(j) = 0$ and $\xi^{u-\langle j\rangle} = 0$. Clearly, $\{x^{(\alpha)}\xi^u \mid \alpha \in \mathcal{O}(m, \underline{t}),\ u \in \mathbb{B}\}$ constitutes an $\mathbb{F}$-basis of $\mathcal{O}(m, n, \underline{t})$.

Let $D_1, \cdots, D_m, d_1, \cdots, d_n$ be linear transformations of $\mathcal{O}(m, n, \underline{t})$ and $\varepsilon_i := (\delta_{i1}, \cdots, \delta_{im})$ such that

$$D_i(x^{(\alpha)}\xi^u) = x^{(\alpha-\varepsilon_i)}\xi^u, \quad i \in Y_0; \qquad d_j(x^{(\alpha)}\xi^u) = (-1)^{u(j)}x^{(\alpha)}\xi^{u-\langle j\rangle}, \quad j \in Y_1,$$

where δ_{ij} is Kronecker delta defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Set

$$W := \left\{ \sum_{i=1}^{m} f_i D_i + \sum_{j=1}^{n} f_j d_j \;\middle|\; f_i,\ f_j \in \mathcal{O}(m, n, \underline{t}),\ i \in Y_0,\ j \in Y_1 \right\}.$$

Then W is the generalized Witt Lie superalgebra over a field $\mathbb{F}$ of prime characteristic, which is contained in $\mathrm{Der}(\mathcal{O}(m, n, \underline{t}))$. In particular, it is a finite-dimensional simple Lie superalgebra (see [14]). Clearly, W possesses a standard $\mathbb{F}$-basis

$$\{x^{(\alpha)}\xi^u D_i,\ x^{(\alpha)}\xi^u d_j \mid \alpha \in \mathcal{O}(m, \underline{t}),\ u \in \mathbb{B},\ i \in Y_0,\ j \in Y_1\}.$$

From this vector space basis, one can read off that

$$W = \mathcal{W} \bigoplus \mathfrak{W},$$

where

$$\mathcal{W} = \bigoplus_{i=0}^{n} \mathcal{W}_i = \bigoplus_{i=0}^{n} \langle x^{(\alpha)}\xi^u D_k \mid \alpha \in \mathcal{O}(m, \underline{t}),\ u \in \mathbb{B},\ |u| = i,\ k \in Y_0\rangle,$$
$$\mathfrak{W} = \bigoplus_{j=0}^{n} \mathfrak{W}_j = \bigoplus_{j=0}^{n} \langle x^{(\alpha)}\xi^u d_l \mid \alpha \in \mathcal{O}(m, \underline{t}),\ u \in \mathbb{B},\ |u| = j,\ l \in Y_1\rangle.$$

A verification shows that $\mathfrak{gl}_{0|2}$ is in fact a Lie algebra which is isomorphic to the subalgebra $\langle \xi_i d_j \mid i, j = 1, 2\rangle$ of W. Then W can be regarded as a $\mathfrak{gl}_{0|2}$-module by means of the adjoint representation. Thus it makes sense to consider the outer derivation $\mathrm{Oder}(\mathfrak{gl}_{0|2}, W)$ of $\mathfrak{gl}_{0|2}$ with coefficients in W.

3 Oder($\mathfrak{gl}_{0|2}$, W) in Prime Characteristic

In this section, we deal with Oder($\mathfrak{gl}_{0|2}$, W) in prime characteristic.

Hereafter, let α be an element of $\mathbb{A}(m, \underline{t})$, k be an element of Y_0, u, v be members of set $\mathbb{B}$ and $\{\xi_i d_j \mid i, j = 1, 2\}$ as a basis of $\mathfrak{gl}_{0|2}$. Since $x^{(\alpha)}$ has no effects under the adjoint operation of $\mathfrak{gl}_{0|2}$, it will be omitted in the following proofs. For any a_i, $b_j \in \mathbb{F}$ and i, $j \in \{1, 2, 3, 4\}$, the matrix is written as follows.

$$\begin{pmatrix} \underbrace{\begin{matrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{matrix}}_{m} & \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} \end{pmatrix}_{4\times(m+1)} := \begin{pmatrix} \underbrace{\begin{matrix} a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & \cdots & a_3 \\ a_4 & a_4 & \cdots & a_4 \end{matrix}}_{m} & \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{matrix} \end{pmatrix}_{4\times(m+1)}.$$

For $i \in Y_1$, it suffices to consider Oder($\mathfrak{gl}_{0|2}$, $\mathcal{W}_i$) and Oder($\mathfrak{gl}_{0|2}$, $\mathfrak{W}_i$). On the one hand, we deal with the outer derivations Oder($\mathfrak{gl}_{0|2}$, $\mathcal{W}_i$).

Proposition 3.1 Oder($\mathfrak{gl}_{0|2}$, $\mathcal{W}_0$) = 0.

Proof. A verification shows that $[x, \mathcal{W}_0] = 0$ for every $x \in \mathfrak{gl}_{0|2}$. If φ is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_0$, then $\varphi = 0$, i.e., Der($\mathfrak{gl}_{0|2}$, $\mathcal{W}_0$) = 0. The proof is completed.

Proposition 3.2 *The matrix* $\boldsymbol{A}$ *is a matrix representation of* Oder($\mathfrak{gl}_{0|2}$, $\mathcal{W}_1$) *of* $\mathfrak{gl}_{0|2}$ *with coefficients in* $\mathcal{W}_1$, *where*

$$\boldsymbol{A} = \begin{pmatrix} \underbrace{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}}_{m} & \underbrace{\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix}}_{m} & \underbrace{\begin{matrix} 0 \\ 0 \\ c_3 \\ c_3 \end{matrix}}_{m} & \underbrace{\begin{matrix} 0 \\ 0 \\ c_4 \\ c_4 \end{matrix}}_{m} & \begin{matrix} \cdots \\ \cdots \\ \cdots \\ \cdots \end{matrix} & \underbrace{\begin{matrix} 0 \\ 0 \\ c_n \\ c_n \end{matrix}}_{m} \end{pmatrix}_{4\times nm}, \qquad c_3, c_4, \cdots, c_n \in \mathbb{F}.$$

Proof. Recall that $\mathcal{W}_1 = \langle \xi_i D_k \rangle$ for $i \in Y_1$. It suffices to consider Der($\mathfrak{gl}_{0|2}$, $\langle \xi_i D_k \rangle$) and Ider($\mathfrak{gl}_{0|2}$, $\langle \xi_i D_k \rangle$). Since $[x, \xi_i D_1]$, $[x, \xi_i D_2]$, $\cdots$, $[x, \xi_i D_m]$ are all equal to $[x, \xi_i D_k]$ for $x \in \mathfrak{gl}_{0|2}$, we denote $\xi_i D_1, \xi_i D_2, \cdots, \xi_i D_m$ by $\xi_i D_k$, $i \in Y_1$.

(i) The derivation space from $\mathfrak{gl}_{0|2}$ into $\langle \xi_i D_k \mid i \in Y_1 \rangle$.

Suppose that

$$\varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ c_{31} & c_{32} & \cdots & c_{3n} \\ c_{41} & c_{42} & \cdots & c_{4n} \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix}$$

is a derivation from $\mathfrak{gl}_{0|2}$ into $\langle \xi_i D_k \mid i \in Y_1 \rangle$, where $c_{ij} \in \mathbb{F}$, $i \in \{1, 2, 3, 4\}$, $j \in Y_1$. By replacing x and y with the standard basis of $\mathfrak{gl}_{0|2}$ in Eq. (2.1), we have

$$\varphi([\xi_1 d_2, \xi_2 d_1]) = [\xi_1 d_2, \varphi(\xi_2 d_1)] - [\xi_2 d_1, \varphi(\xi_1 d_2)] = \varphi(\xi_1 d_1 - \xi_2 d_2), \tag{3.1}$$

$$\varphi([\xi_1 d_2, \xi_1 d_1]) = [\xi_1 d_2, \varphi(\xi_1 d_1)] - [\xi_1 d_1, \varphi(\xi_1 d_2)] = -\varphi(\xi_1 d_2), \tag{3.2}$$

$$\varphi([\xi_1 d_2,\, \xi_2 d_2]) = [\xi_1 d_2,\, \varphi(\xi_2 d_2)] - [\xi_2 d_2,\, \varphi(\xi_1 d_2)] = \varphi(\xi_1 d_2), \tag{3.3}$$

$$\varphi([\xi_2 d_1,\, \xi_1 d_1]) = [\xi_2 d_1,\, \varphi(\xi_1 d_1)] - [\xi_1 d_1,\, \varphi(\xi_2 d_1)] = \varphi(\xi_2 d_1), \tag{3.4}$$

$$\varphi([\xi_2 d_1,\, \xi_2 d_2]) = [\xi_2 d_1,\, \varphi(\xi_2 d_2)] - [\xi_2 d_2,\, \varphi(\xi_2 d_1)] = -\varphi(\xi_2 d_1), \tag{3.5}$$

$$\varphi([\xi_1 d_1,\, \xi_2 d_2]) = [\xi_1 d_1,\, \varphi(\xi_2 d_2)] - [\xi_2 d_2,\, \varphi(\xi_1 d_1)] = 0. \tag{3.6}$$

(3.1) implies that

$$\begin{aligned} &[\xi_1 d_2,\, c_{21}\xi_1 D_k + c_{22}\xi_2 D_k + \cdots + c_{2n}\xi_n D_k] - \\ &[\xi_2 d_1, c_{11}\xi_1 D_k + c_{12}\xi_2 D_k + \cdots + c_{1n}\xi_n D_k] \\ = {} & (c_{31} - c_{41})\xi_1 D_k + (c_{32} - c_{42})\xi_2 D_k + \cdots + (c_{3n} - c_{4n})\xi_n D_k \\ = {} & c_{22}\xi_1 D_k - c_{11}\xi_2 D_k. \end{aligned}$$

A comparison of coefficients yields

$$\begin{cases} c_{22} = c_{31} - c_{41}, \\ c_{11} = c_{42} - c_{32}, \\ c_{33} = c_{43}, \\ \quad \vdots \\ c_{3n} = c_{4n}. \end{cases}$$

Analogously, (3.2)–(3.6) show that

$$\begin{cases} c_{32} = c_{12} = \cdots = c_{1n} = 0, \\ c_{21} = c_{23} = \cdots = c_{2n} = 0, \\ c_{41} = c_{32} = 0, \\ c_{31} = c_{22}, \\ c_{42} = c_{11}. \end{cases}$$

Therefore,

$$\varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & 0 & 0 & \cdots & 0 \\ 0 & c_{22} & 0 & \cdots & 0 \\ c_{22} & 0 & c_{33} & \cdots & c_{3n} \\ 0 & c_{11} & c_{33} & \cdots & c_{3n} \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix},$$

where c_{11}, c_{22}, c_{33}, $\cdots$, $c_{3n} \in \mathbb{F}$.

(ii) The inner derivation space from $\mathfrak{gl}_{0|2}$ into $\langle \xi_i D_k \mid i \in Y_1 \rangle$.

By the definition of inner derivation, elements of $\{\psi_{\xi_i D_k} \mid i \in Y_1\}$ are as follows:

$$\psi_{\xi_1 D_k} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix},$$

$$\psi_{\xi_2 D_k} \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \xi_1 D_k \\ \xi_2 D_k \\ \vdots \\ \xi_n D_k \end{pmatrix},$$

$$\psi_{\xi_i D_k}\begin{pmatrix}\xi_1 d_2\\ \xi_2 d_1\\ \xi_1 d_1\\ \xi_2 d_2\end{pmatrix}=\begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\0&0&\cdots&0\\0&0&\cdots&0\end{pmatrix}\begin{pmatrix}\xi_1 D_k\\ \xi_2 D_k\\ \vdots\\ \xi_n D_k\end{pmatrix},\qquad i\in Y_1\setminus\{1,\,2\}.$$

Hence

$$\psi_{\xi_i D_k}\begin{pmatrix}\xi_1 d_2\\ \xi_2 d_1\\ \xi_1 d_1\\ \xi_2 d_2\end{pmatrix}=\begin{pmatrix}c_1&0&0&\cdots&0\\0&c_2&0&\cdots&0\\c_2&0&0&\cdots&0\\0&c_1&0&\cdots&0\end{pmatrix}\begin{pmatrix}\xi_1 D_k\\ \xi_2 D_k\\ \vdots\\ \xi_n D_k\end{pmatrix},\qquad c_1,\,c_2\in\mathbb{F},\ i\in Y_1.$$

Combining (i) and (ii), the proof is completed.

Proposition 3.3 $\mathrm{Oder}(\mathfrak{gl}_{0|2},\ \mathcal{W}_i)=0$, $i\in Y_1$.

Proof. Recall that $\mathcal{W}_2=\langle \xi_i\xi_j D_k \mid i,j\in Y_1,\ i<j\rangle$. In the case of $i,j\in Y_1\setminus\{1,2\}$, we have $[x,\,\mathcal{W}_2]=0$ for any $x\in\mathfrak{gl}_{0|2}$. Suppose that φ is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_2$. Then we can easily prove $\varphi=0$. For $i\in\{1,\,2\}$, $\mathrm{Der}(\mathfrak{gl}_{0|2},\,\mathcal{W}_2)$ and $\mathrm{Ider}(\mathfrak{gl}_{0|2},\,\mathcal{W}_2)$ are considered, respectively.

Suppose that

$$\varphi\begin{pmatrix}\xi_1 d_2\\ \xi_2 d_1\\ \xi_1 d_1\\ \xi_2 d_2\end{pmatrix}=\begin{pmatrix}c_{12}&\cdots&c_{1n}&b_{13}&\cdots&b_{1n}\\c_{22}&\cdots&c_{2n}&b_{23}&\cdots&b_{2n}\\c_{32}&\cdots&c_{3n}&b_{33}&\cdots&b_{3n}\\c_{42}&\cdots&c_{4n}&b_{43}&\cdots&b_{4n}\end{pmatrix}\begin{pmatrix}\xi_1\xi_2 D_k\\ \vdots\\ \xi_1\xi_n D_k\\ \xi_2\xi_3 D_k\\ \vdots\\ \xi_2\xi_n D_k\end{pmatrix}$$

is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_2$, where $c_{ij},\,b_{it}\in\mathbb{F}$, $i\in\{1,\,2,\,3,\,4\}$, $j\in Y_1\setminus\{1\}$ and $t\in Y_1\setminus\{1,\,2\}$. As

$$\varphi([x,\,y])=[x,\,\varphi(y)]-[y,\,\varphi(x)],\qquad x,\,y\in\mathfrak{gl}_{0|2},$$

we have (3.1)–(3.6). By a comparison of coefficients, we have

$$\begin{cases}c_{12}=c_{43}=\cdots=c_{4n}=0,\\ c_{22}=\cdots=c_{2n}=0,\\ b_{33}=\cdots=b_{3n}=0,\\ b_{13}=\cdots=b_{1n}=0,\\ c_{32}=c_{42},\\ c_{3t}=b_{2t},\\ c_{1t}=b_{4t}.\end{cases}$$

Then

$\mathrm{Der}(\mathfrak{gl}_{0|2},\,\mathcal{W}_2)$

$$= \left\{ \varphi\colon \varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & c_{13} & \cdots & c_{1n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & c_{33} & \cdots & c_{3n} \\ c_{32} & c_{33} & \cdots & c_{3n} & 0 & \cdots & 0 \\ c_{32} & 0 & \cdots & 0 & c_{13} & \cdots & c_{1n} \end{pmatrix} \begin{pmatrix} \xi_1\xi_2 D_k \\ \vdots \\ \xi_1\xi_n D_k \\ \xi_2\xi_3 D_k \\ \vdots \\ \xi_2\xi_n D_k \end{pmatrix} \right\},$$

where $c_{13}, \cdots, c_{1n}, c_{32}, \cdots, c_{3n} \in \mathbb{F}$.

Using the methods in the proof of Proposition 3.2, we obtain inner derivation space from $\mathfrak{gl}_{0|2}$ into $\mathcal{W}_2$ as follows:

$$\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_2)$$
$$= \left\{ \psi\colon \psi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & c_{13} & \cdots & c_{1n} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & c_{33} & \cdots & c_{3n} \\ c_{32} & c_{33} & \cdots & c_{3n} & 0 & \cdots & 0 \\ c_{32} & 0 & \cdots & 0 & c_{13} & \cdots & c_{1n} \end{pmatrix} \begin{pmatrix} \xi_1\xi_2 D_k \\ \vdots \\ \xi_1\xi_n D_k \\ \xi_2\xi_3 D_k \\ \vdots \\ \xi_2\xi_n D_k \end{pmatrix} \right\},$$

where $c_{13}, \cdots, c_{1n}, c_{32}, \cdots, c_{3n} \in \mathbb{F}$. Above all, they show that

$$\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathcal{W}_2) = \mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathcal{W}_2),$$

so

$$\mathrm{Oder}(\mathfrak{gl}_{0|2}, \mathcal{W}_2) = 0.$$

It follows that $\mathrm{Oder}(\mathfrak{gl}_{0|2}, \mathcal{W}_i) = 0$, $i \in Y_1$ from induction on i.

Now the following theorem is a direct consequence of (2.2) and Propositions 3.1–3.3.

Theorem 3.1 *The matrix* $\boldsymbol{A}$ *determines the outer derivation* $\mathrm{Oder}(\mathfrak{gl}_{0|2}, \mathcal{W})$ *of* $\mathfrak{gl}_{0|2}$ *with coefficients in* $\mathcal{W}$, *where*

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ \underbrace{0}_{m} & \underbrace{0}_{m} & \underbrace{c_3}_{m} & \underbrace{c_4}_{m} & \cdots & \underbrace{c_n}_{m} \end{pmatrix}_{4\times nm}, \qquad c_3, c_4, \cdots, c_n \in \mathbb{F}.$$

On the other hand, the outer derivations $\mathrm{Oder}(\mathfrak{gl}_{0|2}, \mathfrak{W}_i)$ for all $i \in Y_1$ are calculated. Similarly, by methods employed in Proposition 3.2, the following results may be verified.

Proposition 3.4 *The matrix* $\boldsymbol{B}$ *is a matrix representation of* $\mathrm{Oder}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)$, *where*

$$\boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \\ 0 & 0 & c_3 & c_4 & \cdots & c_n \end{pmatrix}_{4\times n}, \qquad c_3, c_4, \cdots, c_n \in \mathbb{F}.$$

Proof. Applying the methods of Proposition 3.2, we consider $\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)$ and $\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0)$, where $\mathfrak{W}_0 = \langle d_l \mid l \in Y_1 \rangle$.

(i) The derivation space from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_0$.

Suppose that

$$\varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ c_{31} & c_{32} & \cdots & c_{3n} \\ c_{41} & c_{42} & \cdots & c_{4n} \end{pmatrix}_{4\times n} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

is a derivation from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_0$, where $c_{ij} \in \mathbb{F}$, $i \in \{1, 2, 3, 4\}$, $j \in Y_1$. For any $x, y \in \mathfrak{gl}_{0|2}$, the identity $\varphi([x, y]) = [x, \varphi(y)] - [y, \varphi(x)]$ yields (3.1)–(3.6). A comparison of coefficients shows that

$$\begin{cases} c_{32} = c_{11} = c_{13} = \cdots = c_{1n} = 0, \\ c_{41} = c_{22} = c_{23} = \cdots = c_{2n} = 0, \\ c_{12} = c_{31}, \\ c_{21} = c_{42}, \\ c_{3t} = c_{4t}, \end{cases}$$

where $t \in Y_1 \setminus \{1, 2\}$. Then

$$\mathrm{Der}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0) = \left\{ \varphi\colon \varphi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{pmatrix} 0 & c_{12} & 0 & \cdots & 0 \\ c_{21} & 0 & 0 & \cdots & 0 \\ c_{12} & 0 & c_{33} & \cdots & c_{3n} \\ 0 & c_{21} & c_{33} & \cdots & c_{3n} \end{pmatrix}_{4\times n} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \right\},$$

where $c_{12}, c_{21}, c_{33}, \cdots, c_{3n} \in \mathbb{F}$.

(ii) The inner derivation space from $\mathfrak{gl}_{0|2}$ into $\mathfrak{W}_0$.

Using the methods in the proof of Proposition 3.2, we obtain that

$$\mathrm{Ider}(\mathfrak{gl}_{0|2}, \mathfrak{W}_0) = \left\{ \psi\colon \psi \begin{pmatrix} \xi_1 d_2 \\ \xi_2 d_1 \\ \xi_1 d_1 \\ \xi_2 d_2 \end{pmatrix} = \begin{bmatrix} 0 & a & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & 0 \\ a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \end{bmatrix}_{4\times n} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \right\},$$

where $a, b \in \mathbb{F}$.

Combining (i) and (ii), the result follows.

By similar methods as in Propositions 3.2–3.4, we have

Proposition 3.5 *The matrix $\boldsymbol{C}$ is a matrix representation of* $\mathrm{Oder}(\mathfrak{gl}_{0|2}, \mathfrak{W}_1)$*, where*

$$\boldsymbol{C} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ a & a & c_3 & \cdots & c_n & 0 \\ a & a & \underbrace{c_3}_{n-2} & \cdots & \underbrace{c_n}_{n-2} & \underbrace{0}_{4(n-2)+2} \end{pmatrix}_{4\times n^2}, \qquad a, c_3, \cdots, c_n \in \mathbb{F}.$$

Proposition 3.6 $\mathrm{Oder}(\mathfrak{gl}_{0|2}, \mathfrak{W}_n) = 0.$

Hereafter, we denote the combinatorial number $\dfrac{n!}{m!(n-m)!}$ by C_n^m, where $m \le n$.

Proposition 3.7 *The matrix* $\boldsymbol{D}_2$ *is a matrix representation of* Oder($\mathfrak{gl}_{0|2}$, $\mathfrak{W}_2$) *of* $\mathfrak{gl}_{0|2}$ *with coefficients in* $\mathfrak{W}_2$, *where*

$$\boldsymbol{D}_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ c & c & 0 & \cdots & 0 \\ \underbrace{c}_{C_{n-2}^1} & \underbrace{c}_{C_{n-2}^1} & 0 & \cdots & 0 \end{pmatrix}_{4\times nC_n^2}, \qquad c \in \mathbb{F}.$$

Corollary 3.1 *For* $i \in Y_1 \setminus \{1, n\}$ *and* $c \in \mathbb{F}$, *the matrix* $\boldsymbol{D}_i$ *is a matrix representation of* Oder($\mathfrak{gl}_{0|2}$, $\mathfrak{W}_i$), *where*

$$\boldsymbol{D}_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ c & c & 0 & \cdots & 0 \\ \underbrace{c}_{C_{n-2}^{i-1}} & \underbrace{c}_{C_{n-2}^{i-1}} & 0 & \cdots & 0 \end{pmatrix}_{4\times nC_n^i}.$$

Proof. The result may be proved by induction on i.

Theorem 3.2 *If* $(\boldsymbol{B}|\boldsymbol{C}|\boldsymbol{D}_2|\cdots|\boldsymbol{D}_{n-1})$ *is a block matrix with subblocks* $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}_2$, $\cdots$, $\mathcal{D}_{n-1}$ *as in Propositions* 3.4, 3.5 *and Corollary* 3.1, *then* $(\boldsymbol{B}|\boldsymbol{C}|\boldsymbol{D}_2|\cdots|\boldsymbol{D}_{n-1})$ *is a matrix representation of the outer derivation* Oder($\mathfrak{gl}_{0|2}, \mathfrak{W}$) *of* $\mathfrak{gl}_{0|2}$ *with coefficients in* $\mathfrak{W}$.

Proof. This is a direct consequence of (2.2), Propositions 3.4–3.7 and Corollary 3.1.

Theorem 3.3 *If* $(\boldsymbol{A}|\boldsymbol{B}|\boldsymbol{C}|\boldsymbol{D}_2|\cdots|\mathcal{D}_{n-1})$ *is a block matrix with subblocks* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}_2$, $\cdots$, $\boldsymbol{D}_{n-1}$ *as in Theorem* 3.1, *Propositions* 3.5, 3.5 *and Corollary* 3.1, *then* $(\boldsymbol{A}|\boldsymbol{B}|\boldsymbol{C}|\boldsymbol{D}_2|\cdots|\boldsymbol{D}_{n-1})$ *is a matrix representation of the outer derivation* Oder($\mathfrak{gl}_{0|2}$, W) *of* $\mathfrak{gl}_{0|2}$ *with coefficients in* W.

Proof. This is a direct consequence of (2.2), Theorems 3.1 and 3.2.

4 Oder($\mathfrak{gl}_{0|2}$, W) in Characteristic Zero

Imitating the situation that the generalized Witt Lie superalgebra W over a field $\mathbb{F}$ of characteristic zero (see [15], [16]), the elements of $\mathbb{F}[x_1, \cdots, x_m]$ have no effects under the adjoint operation of $\mathfrak{gl}_{0|2}$. So Oder($\mathfrak{gl}_{0|2}$, W) in characteristic zero can be obtained by the same methods as in prime characteristic. Therefore, we have

Theorem 4.1 *Let* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}_i$, $\mathcal{W}_1$, $\mathfrak{W}_0$, $\mathfrak{W}_1$, $\mathfrak{W}_i$ *as in Theorem* 3.3 *and* $\mathbb{F}$ *be the underlying base field of characteristic zero. Then* $(\boldsymbol{A}|\boldsymbol{B}|\boldsymbol{C}|\boldsymbol{D}_2|\cdots|\boldsymbol{D}_{n-1})$ *is a matrix representation of the outer derivation* Oder($\mathfrak{gl}_{0|2}$, W) *of* $\mathfrak{gl}_{0|2}$ *with coefficients in W.*

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