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Univalent Criteria for Analytic Functions Involving Schwarzian Derivative

HU ZHEN-YONG, WANG QI-HAN, HE LIANG-MIAO AND LONG BO-YONG*
(*School of Mathematical Sciences, Anhui University, Hefei, 230601*)

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Abstract: In this paper, some new criteria for univalence of analytic functions defined in the unit disk in terms of two parameters are presented. Moreover, the related result of Aharonov and Elias (Aharonov D, Elias U. Univalence criteria depending on parameters. *Anal. Math. Phys.*, 2014, **4**(1-2): 23–34) is generalized.

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1 Introduction

An analytic function $f(z)$ is said to be univalent in $\mathbb{D} = \{z, |z| < 1\}$ if it is one-to-one in $\mathbb{D}$. As usual, for some simple analytic functions we may judge easily if it is univalent by definition. In fact, we are often faced with complicated analytic function, and it is hard to determine whether to be univalent. Therefore, judging only by definition is not enough. This allows scholars to explore other univalent criteria. Recently, some new univalent criteria for analytic functions have been established in [1]–[4].

The Schwarzian derivative of a locally univalent analytic function $f(z)$ is defined by

$$S_{f(z)} = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The status of $S_{f(z)}$ in the study of univalence is very important. Some classical univalent criteria over the Schwarzian derivative are introduced in the following. Nehari[5] proved that

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* **Corresponding author**.
E-mail address: huzhenyongad@163.com (Hu Z Y), longboyong@ahu.edu.cn (Long B Y).

if

$$|S_{f(z)}| \leqslant \frac{2}{(1-|z|^2)^2}, \qquad z \in \mathbb{D} \tag{1.1}$$

or

$$|S_{f(z)}| \leqslant \frac{\pi^2}{2}, \qquad z \in \mathbb{D}, \tag{1.2}$$

then $f(z)$ is univalent in $\mathbb{D}$. Later, Hille[6] proved the criterion (1.1) is sharp. Pokornyi[7] stated the criterion

$$|S_{f(z)}| \leqslant \frac{4}{1-|z|^2}, \qquad z \in \mathbb{D}. \tag{1.3}$$

But this proof of the result is due to Nehari[8].

In the spirit of Steinmetz[9], Aharonov[10] defined a result of sharpness of univalent criteria.

Theorem 1.1 *A criterion for univalence of the form $|S_{f(z)}| \leqslant 2p(|z|)$ $(z \in \mathbb{D})$ is sharp, if for an analytic function $f(z)$, the conditions $S_{f(x)} \geqslant 2p(x)$, where $x \in (-1, 1)$, and $S_{f(x)} \neq 2p(x)$ imply that $f(z)$ is not univalent in $\mathbb{D}$.*

Nehari[8] proved the following theorem, which provides a method to establish new results on univalent criteria.

Theorem 1.2 *Suppose that*

(a) *$p(x)$ is a positive and continuous even function for $x \in (-1, 1)$;*

(b) *$p(x)(1-x^2)^2$ is nonincreasing for $x \in (0, 1)$;*

(c) *the differential equation*

$$y''(x) + p(x)y(x) = 0, \qquad x \in (-1, 1) \tag{1.4}$$

has a solution which does not vanish in $-1 < x < 1$. Then, any analytic function $f(z)$ in $\mathbb{D}$ satisfying $|S_{f(z)}| \leqslant 2p(|z|)$ is univalent in $\mathbb{D}$.

In view of Theorem 1.2, the univalent criteria (1.1), (1.2) and (1.3) can be given by

$$y(x) = \sqrt{1-x^2}, \qquad y(x) = \cos\frac{\pi x}{2}, \qquad y(x) = 1 - x^2,$$

respectively.

Let $p(z)$ be analytic in $\mathbb{D}$ and consider the analytic differential equation

$$y''(z) + p(z)y(z) = 0. \tag{1.5}$$

Further, let $u(z)$ and $v(z)$ be two linearly independent functions (solutions of (1.5)). Under the assumptions of Theorem 1.2, if $p(z)$ is self majorant (an analytic function $p(z)$ in the open $\mathbb{D}$ is said to be self majorant, if $|p(z)| \leqslant p(|z|)$ for all $z \in \mathbb{D}$), then

$$f_0(z) = \frac{v(z)}{u(z)}$$

satisfies

$$|S_{f_0(z)}| = 2|p(z)| \leqslant 2p(|z|),$$

and $f_0(z)$ is univalent. Noting that

$$\frac{\partial f_0(z)}{\partial z} = \frac{v'(z)u(z) - v(z)u'(z)}{u(z)^2} \tag{1.6}$$

and

$$W[u(z),\, v(z)] = \begin{vmatrix} u(z) & v(z) \\ u'(z) & v'(z) \end{vmatrix} = v'(z)u(z) - v(z)u'(z) = c_1 \neq 0,$$

where $W[u(z),\, v(z)]$ is the Wronskian determinant of $u(z)$ and $v(z)$, c_1 is a constant. Thus, we rewrite (1.6) as follows

$$\frac{\partial f_0(z)}{\partial z} = c_1 \frac{1}{u(z)^2}. \tag{1.7}$$

Integrating on both sides of (1.7), we obtain

$$f_0(z) = \frac{v(z)}{u(z)} = c_1 \int_0^z \frac{\mathrm{d}t}{u(t)^2} + c_2,$$

where c_1, c_2 are constants. Thus, a special univalent function

$$f_1(z) = \int_0^z \frac{\mathrm{d}t}{u(t)^2}$$

is concluded.

Moreover, Steinmetz[9] proved that if $p(z)$ is self majorant and

$$\int_0^1 \frac{\mathrm{d}t}{y^2(t)} = \infty, \tag{1.8}$$

where $y(x)$ is given by (1.4), then the criterion $|S_{f(z)}| \leqslant 2p(|z|)$ is sharp.

Nehari[11] proved a general condition for univalence of analytic functions in $\mathbb{D}$ as follows.

$$|S_{f(z)}| \leqslant \frac{2(1-\mu^2)}{(1-|z|^2)^2} + \frac{2\mu(2+\mu)}{(1+|z|^2)^2}, \qquad 0 \leqslant \mu \leqslant 1.$$

The criterion can be generated by the function

$$y(x) = \frac{(1-x^2)^{\frac{\mu+1}{2}}}{(1+x^2)^{\frac{\mu}{2}}}.$$

As to Schwarzian derivative of analytic functions, we refer to [12]–[15] for more details.

In the next section, we want to find a general $p(x)$. In view of this point, we consider that

$$y(x) = (1-x^2)^a Q(x,\, k) = 0 \qquad \text{at } x = \pm 1,$$

where $y(x)$ is given by (1.4), $a \in \left[\frac{1}{2},\, 1\right]$, k is a parameter. In addition, for $x \in (-1,\, 1)$, $Q(x,\, k) \neq 0$ and is even, $Q''(x,\, k)$ is continuous for $x \in [-1,\, 1]$. According to the fundamental conditions, we give a genernal $p(x)$. Furthermore, we present some special $p(x)$.

2 Main Results

Theorem 2.1 *Let*

$$p(x)=\frac{-(1-x^2)^2Q''(x,\,k)+4ax(1-x^2)Q'(x,\,k)+2a[1+(1-2a)x^2]Q(x,\,k)}{(1-x^2)^2Q(x,\,k)}, \qquad (2.1)$$

where $a\in\left[\frac{1}{2},\,1\right]$, *$k$ is a parameter with*

(i) *For* $x\in(-1,\,1)$, $Q(x,\,k)\neq 0$ *and is even;*

(ii) $Q''(x,\,k)$ *is continuous for* $x\in[-1,\,1]$;

(iii) $p(z)$ *is self majorant, where* $z\in\mathbb{D}$, *and* $p(x)(1-x^2)^2$ *is nonincreasing for* $x\in(0,\,1)$. *Furthermore, if* $f(z)$ *is an analytic function in* $\mathbb{D}$ *satisfying* $|S_{f(z)}|\leqslant 2p(|z|)$, *then* $f(z)$ *is univalent in* $\mathbb{D}$, *and the result is sharp.*

Proof. Let

$$y(x)=(1-x^2)^aQ(x,\,k),$$

where $a\in\left[\frac{1}{2},\,1\right]$. Then for $x\in(-1,\,1)$, by $Q(x,\,k)\neq 0$, we have

$$y(x)\neq 0.$$

A tedious calculation yields (2.1) from the differential equation (1.4).

Let

$$\varphi(x)=p(x)(1-x^2)^2.$$

Then, under the condition (ii), it is easy to see that

$$\varphi(1)=4a(1-a)\geq 0, \qquad a\in\left[\frac{1}{2},\,1\right].$$

Combining (iii), we know that

$$\varphi(x)>0, \qquad x\in(-1,\,1).$$

This means that

$$p(x)>0, \qquad x\in(-1,\,1).$$

In addition, for $x\in(-1,\,1)$, by $Q(x,\,k)\neq 0$ and (ii), it follows that $p(x)$ is continuous for $x\in(-1,\,1)$. Obviously, from (2.1), we can get that $Q(x,\,k)$ is even for $x\in(-1,\,1)$ implies that $p(x)$ is even for $x\in(-1,\,1)$. By (iii) again, according to Theorem 1.2, we know that $f(z)$ is univalent in $\mathbb{D}$.

Next, we prove the sharpness of the theorem.

For $x\in(-1,\,1)$, by (ii) and $Q(x,\,k)\neq 0$, we have that there exists an $M>0$ such that

$$0<|Q(x,\,k)|<M.$$

Thus, combining $a\in\left[\frac{1}{2},\,1\right]$, we have

$$\begin{aligned}\int\frac{\mathrm{d}t}{y^2(t)}&=\int\frac{\mathrm{d}t}{(1-t^2)^{2a}Q^2(t,\,k)}\\&>\frac{1}{M^2}\int\frac{\mathrm{d}t}{1-t^2}\\&=\frac{1}{2M^2}\ln\left|\frac{t+1}{t-1}\right|+c\end{aligned}$$

is divergent at $t = 1$, and then (1.8) is obvious. Combing $p(z)$ is self majorant, where $z \in \mathbb{D}$, this completes the proof of theorem.

Note that $p(x)$ is abstract in Theorem 2.1. The following results give some concrete $p(x)$.

Theorem 2.2 *Let*

$$p(x) = \frac{-4a(a-1)x^2}{(1-x^2)^2} + \frac{2a+2k-2k(1+5a)x^2}{(1-kx^2)(1-x^2)}, \tag{2.2}$$

where

$$a \in \left[\frac{1}{2},\ 1\right], \qquad k \in \left[1+2a-\sqrt{6a^2+3a+1},\ \frac{2a-1}{2a+3}\right]. \tag{2.3}$$

If $f(z)$ is an analytic function in $\mathbb{D}$ satisfying

$$|S_{f(z)}| \leqslant 2p(|z|),$$

then $f(z)$ is univalent in $\mathbb{D}$. Moreover, the result is sharp.

Proof. Let

$$Q(x,\, k) = 1 - kx^2, \qquad k < 1,\ x \in (-1,\, 1).$$

Then $Q(x, k)$ satisfies the conditions (i) and (ii) of Theorem 2.1. Thus, applying (2.1), we can get (2.2). A straightforward calculation gives

$$p(x)(1-x^2)^2 = -4a(a-1)x^2 + \frac{[2a+2k-2k(1+5a)x^2](1-x^2)}{1-kx^2}. \tag{2.4}$$

Using $x^2 = t$, then (2.4) is equivalent to the following

$$\psi(t) :\equiv -4a(a-1)t + \frac{[2a+2k-2k(1+5a)t](1-t)}{1-kt},$$

it follows

$$\psi'(t) = \frac{-2k^2(1+3a+2a^2)t^2 + 4k(1+3a+2a^2)t + 2k^2 - 4k - 8ak + 2a - 4a^2}{(1-kt)^2}.$$

Let

$$\sigma(t) = -2k^2(1+3a+2a^2)t^2 + 4k(1+3a+2a^2)t + 2k^2 - 4k - 8ak + 2a - 4a^2. \tag{2.5}$$

In order to apply Theorem 2.1, we firstly show that

$$\sigma(t) \le 0, \qquad t \in (0,\, 1). \tag{2.6}$$

When $k = 0$, $a \in \left[\frac{1}{2},\ 1\right]$. It follows that

$$\sigma(t) = 2a(1-2a) \le 0.$$

When $k \neq 0$, note that

$$1 + 3a + 2a^2 > 0, \qquad a \in \left[\frac{1}{2},\ 1\right].$$

According to the graph of equation (2.5), if $0 < k < 1$, then the symmetric axis $t = \frac{1}{k} > 1$ of the graph of equation (2.5). Obviously,

$$\sigma(1) = (-4a^2 - 6a)k^2 + (8a^2 + 4a)k + 2a - 4a^2 \le 0$$

means that (2.6) holds. It follows

$$k \in \left(0,\ \frac{2a-1}{2a+3}\right].$$

If $k < 0$, then the symmetric axis $t = \dfrac{1}{k} < 0$ of the graph of equation (2.5), and

$$\sigma(0) = 2k^2 - (4+8a)k + 2a - 4a^2 \le 0$$

means that (2.6) holds. It follows

$$k \in [1 + 2a - \sqrt{6a^2+3a+1},\ 0).$$

Thus, we obtain (2.3) from above analysis.

Next, we need to prove $p(z)$ is self majorant.

In fact, it is obvious from the proof of Example 2.1. This completes the proof of theorem.

Remark 2.1 According to Theorem 2.2, when $a = 1$, we can get that

$$k \in \left[3 - \sqrt{10},\ \frac{1}{5}\right].$$

Thus Theorem 2.2 reduces to Theorem 1 of [10].

Example 2.1 Let

$$f(z) = \int_0^z \frac{\mathrm{d}t}{(1-t^2)^{2a}(1-kt^2)^2}, \qquad z \in \mathbb{D}$$

satisfy the conditions $a \in \left[\frac{1}{2},\ 1\right]$ and $k \in \left[1 + 2a - \sqrt{6a^2+3a+1},\ \frac{2a-1}{2a+3}\right]$. Then $f(z)$ is univalent in $\mathbb{D}$.

Proof. We rewrite (2.2) as

$$p(x) = \frac{-4a(a-1)x^2}{(1-x^2)^2} + \frac{2a+2k}{1-kx^2} + \frac{2a(1-5k)x^2}{(1-kx^2)(1-x^2)}.$$

According to the conditions, when $k \in \left[0,\ \frac{2a-1}{2a+3}\right]$ and $a \in \left[\frac{1}{2},\ 1\right]$, we have

$$2a + 2k > 0, \qquad 1 - 5k \geqslant 1 - 5 \times \frac{2a-1}{2a+3} \geqslant 0.$$

When $k \in \left[1 + 2a - \sqrt{6a^2+3a+1},\ 0\right)$ and $a \in \left[\frac{1}{2},\ 1\right]$, we have

$$2a + 2k \geqslant 2(1 + 3a - \sqrt{6a^2+3a+1}) > 0, \qquad 1 - 5k > 0.$$

This means that all Taylor coefficients of $p(z)$ are positive. Thus,

$$|S_{f(z)}| = 2|p(z)| \leqslant 2p(|z|),$$

and the rest follows from Theorem 2.2.

Although the following result is also obtained by using Theorem 2.1, here we do not consider its sharpness.

Theorem 2.3 *Let*

$$p(x) = \frac{-4a(a-1)x^2}{(1-x^2)^2} + \frac{4a-2k+(2a+2k+8ak)x^2}{(2+x^2)(1-x^2)} - \frac{4k(k-1)x^2}{(2+x^2)^2}, \tag{2.7}$$

where $a \in \left[\frac{1}{2},\ 1\right]$ *and* $k \in \left[\frac{3-6a}{4},\ \frac{1-2a}{2}\right]$. *If* $f(z)$ *is an analytic function in* $\mathbb{D}$ *satisfying* $|S_{f(z)}| \leqslant 2p(|z|)$, *then* $f(z)$ *is univalent in* $\mathbb{D}$.

Proof. Choosing $Q(x,\ k) = (2+x^2)^k$, we can obtain (2.7) easily from (2.1). Let $x^2 = t$. Then a direct computation gives

$$\begin{aligned}\eta(t) &\equiv: p(t)(1-t)^2 \\ &= -4a(a-1)t + \frac{[4a-2k+(2a+2k+8ak)t](1-t)}{2+t} - \frac{4k(k-1)t(1-t)^2}{(2+t)^2},\end{aligned}$$

it follows that

$$\begin{aligned}\eta'(t) = &\frac{(2a-4a^2-8ak-4k^2+2k)t^3+12(a-2a^2-4ak-2k^2+k)t^2}{(2+t)^3} \\ &+ \frac{(24a-48a^2-48ak+36k^2-42k)t+16a-32a^2+32ak-8k^2+28k}{(2+t)^3}.\end{aligned}$$

Let

$$\begin{aligned}\chi(t) = &(2a-4a^2-8ak-4k^2+2k)t^3+12(a-2a^2-4ak-2k^2+k)t^2 \\ &+(24a-48a^2-48ak+36k^2-42k)t+16a-32a^2+32ak-8k^2+28k. \end{aligned}\tag{2.8}$$

Applying Theorem 2.1, we have to prove that

$$\eta'(t) \leqslant 0, \qquad t \in (0,\ 1).$$

We first consider the following three necessary conditions:

$$p(0) = 2a-k > 0,$$

$$\chi(1) = 54a-108a^2-72ak \leqslant 0, \tag{2.9}$$

$$\chi(0) = 16a-32a^2+32ak-8k^2+28k \leqslant 0. \tag{2.10}$$

It follows that

$$k \in \left[\frac{3-6a}{4},\ \frac{8a+7-\sqrt{144a+49}}{4}\right]. \tag{2.11}$$

From (2.8), we calculate

$$\begin{aligned}\chi''(t) &= 6(2a-4a^2-8ak-4k^2+2k)t+12(2a-4a^2-8ak-4k^2+2k) \\ &= 6(2a-4a^2-8ak-4k^2+2k)(t+2).\end{aligned}$$

Using (2.9) and (2.10), if $\chi(t)$ is convex in $t \in (0,\ 1)$, i.e., $\chi''(t) \geqslant 0$, then it is enough to see that

$$\eta'(t) \leq 0, \qquad t \in (0,\ 1).$$

To make $\chi(t)$ convex. Now we only need

$$2a-4a^2-8ak-4k^2+2k \geqslant 0,$$

which is equivalent to

$$k \in \left[-a,\ \frac{1-2a}{2}\right].$$

In view of (2.11), we conclude that

$$k \in \left[\frac{3-6a}{4},\ \frac{1-2a}{2}\right].$$

This completes the proof of theorem.

Remark 2.2 We add another parameter to $Q(x, k)$, i.e., we consider $L(x, A, B)$, where A, B are positive parameters. If we write

$$L(x, A, B) = A - Bx^2 = A\left(1 - \frac{B}{A}x^2\right),$$

where $A > 0$, in specially, let $k = \dfrac{B}{A}$, then it is equivalent to the case of Theorem 2.2. Moreover, if we let

$$L(x, k, A) = (A + x^2)^k,$$

then it is Theorem 2.1 of [16] when $A = 1$. While, in Theorem 2.3 we consider another case when $A = 2$. In fact, it is interesting to consider the cases for fixing every $A > 0$.

References

[1] Deniz E, Orhan H. Univalence criterion for meromorphic functions and Loewner chains. *Appl. Math. Comput.*, 2011, **218**(3): 751–755.

[2] Deniz E, Orhan H. Univalence criterion for analytic functions. *Gen. Math.*, 2009, **17**(4): 211–220.

[3] Răducanu D. A univalence criterion for analytic functions in the unit disk. *Mathemathica*, 2004, **2**(2): 213–216.

[4] Obradovic M, Ponnusamy S, Tuneski N. Radius of univalence of certain combination of univalent and analytic functions. *Bull. Malays. Math. Sci. Soc.*, 2012, **35**(2): 325–334.

[5] Nehari Z. The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.*, 1949, **55**(6): 545–551.

[6] Hille E. Remarks on a paper by Zeev Nehari. *Bull. Amer. Math. Soc.*, 1949, **55**(11): 552–553.

[7] Pokornyi V. On some sufficient conditions for univalence. *Doklady Akad. Nauk SSSR (N.S.)*, 1951, **79**: 743–746.

[8] Nehari Z. Some criteria of univalence. *Proc. Amer. Math. Soc.*, 1954, **5**(5): 700–704.

[9] Steinmetz N. Homeomorphic extensions of univalent functions. *Complex Var. Elliptic Equ.*, 1986, **6**(1): 1–9.

[10] Aharonov D, Elias U. Univalence criteria depending on parameters. *Anal. Math. Phys.*, 2014, **4**(1-2): 23–34.

[11] Nehari Z. Univalence criteria depending on the Schwarzian derivative. *Illinois J. Math.*, 1979, **23**(3): 345–351.

[12] Nikola T, Biljana J, Boljan P. On existence of sharp univalence criterion using the Schwarzian derivative. *C. R. Acad. Bulgare Sci.*, 2015, **68**(5): 569–576.

[13] Gehring F, Prommerenke C. On the Nehari univalence criterion and quasicircles. *Comment. Math. Helve.*, 1984, **59**(1): 226–242.

[14] Nunokawa M, Uyanik N, Owa S. Univalency of analytic functions associated with Schwarzian derivative. *Acta Univ. Apulensis Math. Inform.*, 2012, **31**(31): 157–161.

[15] Ahmed E, Bakhit M. Characterizations involving Schwarzian derivative in some analytic function spaces. *Math. Sci.*, 2013, **7**(1): 1–8.

[16] Aharonov D, Elias U. Sufficient conditions for univalence of analytic functions, *J. Anal.*, 2014, **22**: 1–11.