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Third Hankel Determinant for the Inverse of Starlike and Convex Functions

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Abstract: Denote $\mathcal{S}$ to be the class of functions which are analytic, normalized and univalent in the open unit disk $\mathbb{U} = \{z\colon |z| < 1\}$. The important subclasses of $\mathcal{S}$ are the class of starlike and convex functions, which we denote by $\mathcal{S}^*$ and $\mathcal{C}$. In this paper, we obtain the third Hankel determinant for the inverse of functions $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ belonging to $\mathcal{S}^*$ and $\mathcal{C}$.

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1 Introduction

Let $\mathcal{H}(\mathbb{U})$ denote the class of functions which are analytic in the open unit disk $\mathbb{U} = \{z\colon |z| < 1\}$. Let $\mathcal{A}$ be the class of all functions $f \in \mathcal{H}(\mathbb{U})$ which are normalized by $f(0) = 0$ and $f'(0) = 1$ and have the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad z \in \mathbb{U}. \tag{1.1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$.

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In [1] and [2], the q-th Hankel determinant for $q \geqslant 1$ and $n \geqslant 1$ is stated by Pommerenke as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $n,\, q \in \mathbf{N}_+$.

Following Pommerenke, many authors focused on the investigating of the second Hankel determinant $H_2(2) = a_2a_4 - a_3^2$ (see [3]–[6]). Only a few papers have been devoted to the third Hankel determinant (see [7]–[11])

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{1.2}$$

We seek upper bound on the third Hankel determinant for the inverse of the classes $\mathcal{S}^*$ of starlike functions and $\mathcal{C}$ of convex functions. The class $\mathcal{S}^*$ and $\mathcal{C}$ are defined as follows.

Definition 1.1 *Let f be given by* (1.1). *Then $f \in \mathcal{S}^*$ if and only if*

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \qquad z \in \mathbb{U}. \tag{1.3}$$

Definition 1.2 *Let f be given by* (1.1). *Then $f \in \mathcal{C}$ if and only if*

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \qquad z \in \mathbb{U}. \tag{1.4}$$

Let $\mathcal{P}$ be the class of all function $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$. The function $p \in \mathcal{P}$ have the following form:

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots, \qquad z \in \mathbb{U}. \tag{1.5}$$

In [7], it was proved that

Theorem 1.1

$$f \in \mathcal{S}^* \Rightarrow |H_3(1)| \leqslant 1,$$

$$f \in \mathcal{C} \Rightarrow |H_3(1)| \leqslant \frac{49}{540} = 0.090\cdots.$$

Lemma 1.1[12] *If $p \in \mathcal{P}$, then the sharp estimate $|p_n| \leqslant 2$ holds for $n = 1, 2, \cdots$.*

Lemma 1.2[13] *If $p \in \mathcal{P}$, then the following estimates holds for $n,\, k = 1, 2, \cdots,\ n > k$:*

$$|p_n - \lambda p_kp_{n-k}| \leqslant \begin{cases} 2, & 0 \leqslant \lambda \leqslant 1; \\ 2|2\lambda - 1|, & \lambda \geqslant 1. \end{cases}$$

2 Main Results

Theorem 2.1 *If $f \in \mathcal{S}^*$ and $f^{-1}(\omega) = \omega + \sum\limits_{n=2}^{\infty} d_n \omega^n$ is the inverse function of f with $|\omega| < r_0$ where r_0 is greater than the radius of the Koebe domain of the class $f \in \mathcal{S}^*$, then we have*

$$|H_3(1)| \leqslant \frac{17}{9} = 1.888\cdots.$$

Proof. From (1.3), it follow that $f \in \mathcal{S}^*$ can be written in the form

$$\frac{zf'(z)}{f(z)} = p(z), \tag{2.1}$$

where p belongs to the class $\mathcal{P}$.

From (2.1) it follows that

$$\begin{cases} a_2 = p_1, \\ a_3 = \dfrac{p_2}{2} + \dfrac{p_1^2}{2}, \\ a_4 = \dfrac{p_3}{3} + \dfrac{p_2 p_1}{2} + \dfrac{p_1^3}{6}, \\ a_5 = \dfrac{p_4}{4} + \dfrac{p_3 p_1}{3} + \dfrac{p_2 p_1^2}{4} + \dfrac{p_2^2}{8} + \dfrac{p_1^4}{24}. \end{cases} \tag{2.2}$$

As

$$f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} d_n \omega^n \tag{2.3}$$

is the inverse function of f, we have

$$f^{-1}[f(z)] = f[f^{-1}(z)] = z. \tag{2.4}$$

From (2.3) and (2.4), we have

$$f^{-1}\left[z + \sum_{n=2}^{\infty} a_n z^n\right] = z. \tag{2.5}$$

From (2.4) and (2.5), we get

$$\begin{aligned} & z + (a_2 + d_2)z^2 + (a_3 + 2a_2 d_2 + d_3)z^3 + [a_4 + d_2(2a_3 + a_2^2) + 3d_3 a_2 + d_4]z^4 \\ & \quad + [a_5 + 2d_2(a_4 + a_2 a_3) + 3d_3(a_3 + a_2^2) + 4d_4 a_2 + d_5]z^5 + \cdots \\ = {} & z. \end{aligned}$$

By comparing the coefficients of z and z^2, we get

$$\begin{cases} d_2 = -a_2, \\ d_3 = 2a_2^2 - a_3, \\ d_4 = -a_4 + 5a_2 a_3 - 5a_2^3, \\ d_5 = -a_5 + 6a_4 a_2 - 21 a_3 a_2^2 + 3a_3^2 + 14 a_2^4. \end{cases} \tag{2.6}$$

From (2.2) and (2.6), we obtain

$$\begin{cases} d_2 = -p_1, \\ d_3 = -\dfrac{p_2}{2} + \dfrac{3p_1^2}{2}, \\ d_4 = -\dfrac{p_3}{3} + 2p_2p_1 - \dfrac{8p_1^3}{3}, \\ d_5 = -\dfrac{p_4}{4} + \dfrac{5p_3p_1}{3} - \dfrac{25p_2p_1^2}{4} + \dfrac{5p_2^2}{8} + \dfrac{125p_1^4}{24}. \end{cases} \tag{2.7}$$

From (2.7) and (1.2), we get

$$H_3(1) = G(p_1,\, p_2,\, p_3,\, p_4),$$

where

$$\begin{aligned} &G(p_1,\, p_2,\, p_3,\, p_4) \\ =\ &\frac{1}{144}[17p_1^6 - 51p_2p_1^4 + 8p_3p_1^3 + 45p_2^2p_1^2 - 18p_1^2p_4 + 24p_3p_2p_1 - 27p_2^3 + 18p_4p_2 - 16p_3^2] \\ =\ &\frac{1}{144}[10(p_2 - p_1^2)(p_4 - p_2^2) + 8(p_2 - p_1^2)(p_4 - p_3p_1) - 17(p_2 - p_1^2)^3 - 16(p_3 - p_2p_1)^2]. \end{aligned}$$

Using triangle inequality and Lemma 1.2, we get Theorem 2.1.

Theorem 2.2 *If $f \in \mathcal{C}$ and $f^{-1}(\omega) = \omega + \sum\limits_{n=2}^{\infty} d_n\omega^n$ is the inverse function of f with $|\omega| < r_0$ where r_0 is greater than the radius of the Koebe domain of the class $f \in \mathcal{C}$, then we have*

$$|H_3(1)| \leqslant \frac{61}{540} = 0.11296\cdots.$$

Proof. Similar approach as in the proof of Theorem 2.1. From (1.4), it follow that $f \in \mathcal{C}$ can be written in the form

$$1 + \frac{zf''(z)}{f'(z)} = p(z), \tag{2.8}$$

where p belongs to the class $\mathcal{P}$. Equating coefficients in (2.8) yields

$$\begin{cases} a_2 = \dfrac{p_1}{2}, \\ a_3 = \dfrac{p_2}{6} + \dfrac{p_1^2}{6}, \\ a_4 = \dfrac{p_3}{12} + \dfrac{p_1p_2}{8} + \dfrac{p_1^3}{24}, \\ a_5 = \dfrac{p_4}{20} + \dfrac{p_3p_1}{15} + \dfrac{p_2p_1^2}{20} + \dfrac{p_2^2}{40} + \dfrac{p_1^4}{120}. \end{cases} \tag{2.9}$$

From (2.6) and (2.9), we get

$$\begin{cases} d_2 = -\dfrac{p_1}{2}, \\ d_3 = -\dfrac{p_2}{6} + \dfrac{p_1^2}{3}, \\ d_4 = -\dfrac{p_3}{12} + \dfrac{7p_1p_2}{24} - \dfrac{p_1^3}{4}, \\ d_5 = -\dfrac{p_4}{20} + \dfrac{11p_3p_1}{60} - \dfrac{23p_2p_1^2}{60} + \dfrac{7p_2^2}{120} + \dfrac{p_1^4}{5}. \end{cases} \tag{2.10}$$

From (2.10) and (1.2), we get

$$H_3(1) = G(p_1, p_2, p_3, p_4),$$

where

$$\begin{aligned}
&G(p_1, p_2, p_3, p_4)\\
&= \frac{1}{8640}[4p_1^6 - 24p_2p_1^4 + 12p_3p_1^3 + 39p_2^2p_1^2 - 36p_1^2p_4 + 36p_3p_2p_1 - 44p_2^3 + 72p_4p_2 - 60p_3^2]\\
&= \frac{1}{2160}\bigg[-8\left(p_2 - \frac{1}{2}p_1^2\right)^3 + 6p_4(p_2 - p_1^2) + 9p_2(p_4 - p_2^2)\\
&\qquad + 3(p_2 - p_1^2)(p_4 - p_1p_3) - 15p_3\left(p_3 - \frac{4}{5}p_1p_2\right) + 6p_2^2\left(p_2 - \frac{3}{8}p_1^2\right)\bigg].
\end{aligned}$$

As above, it is enough to apply the triangle inequality and Lemmas 1.1 and 1.2.

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