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A Note on the Stability of K-g-frames

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Abstract: In this paper, we present a new stability theorem on the perturbation of K-g-frames by using operator theory methods. The result we obtained improves one corresponding conclusion of other authors.

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1 Introduction

A frame for a Hilbert space was discovered in 1950's by Duffin and Schaeffer[1], which has made great contributions to various fields because of its nice properties, the reader can examine [2]–[8] for background and details of frames. G-frames, proposed by Sun[9], generalize the notion of frames extensively, which possess some distinct properties though they share many similar properties with frames (see [10] and [11]).

A K-frame is an extension of a frame, which emerged in the work on atomic systems for operators due to Găvruţa[12], and the results involved show us that the properties of K-frames are quite different from those for frames owing to the linear bounded operator K (see also [13]–[16]).

The idea of Găvruţa has been applied to the case of g-frames by Xiao *et al.*[17] and thus providing us the concept of K-g-frames, which have already attracted many researchers' interest due to their potential flexibility (see [18]–[21] for example). In this paper we pay attention to the stability of K-g-frames, and the motivation derives from an observation on one stability result for K-g-frames, Theorem 4.1 in [21], recently obtained by Hua and Huang. In the proof the authors asserted that the frame operator of the involved K-g-frame is invertible on the whole space, which plays a key role in their proof to show the lower

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K-g-frame bound condition stated in the theorem. In reality, however, the invertibility of the frame operator of a K-g-frame is absent for the whole space, since a K-g-frame is not necessarily a g-frame (see Example 3.1 for details). The purpose of this paper is to provide an improvement to their result.

Throughout this paper, the notations $\mathscr{H}$ and $\mathscr{K}$ are reserved for two Hilbert spaces, and $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ is used to denote a sequence of closed subspaces of $\mathscr{K}$, where the index set $\mathbb{J}$ is finite or countable. The family of all linear bounded operators from $\mathscr{H}$ to $\mathscr{K}$ is designated as $B(\mathscr{H}, \mathscr{K})$, which is abbreviated to $B(\mathscr{H})$ if $\mathscr{H} = \mathscr{K}$. The notation $\mathscr{R}(\theta)$ designates the range of $\theta \in B(\mathscr{H}, \mathscr{K})$.

Let $\ell^2(\{\mathscr{K}_j\}_{j\in\mathbb{J}})$ be the Hilbert space defined by

$$\ell^2(\{\mathscr{K}_j\}_{j\in\mathbb{J}}) = \left\{\{g_j\}_{j\in\mathbb{J}} \colon g_j \in \mathscr{K}_j,\ j \in \mathbb{J}, \text{ and } \sum_{j\in\mathbb{J}} \|g_j\|^2 < \infty\right\},$$

where the inner product is given by

$$\langle \{f_j\}_{j\in\mathbb{J}}, \{g_j\}_{j\in\mathbb{J}}\rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j\rangle.$$

For a sequence of linear bounded operators $\{\Lambda_j\}_{j\in\mathbb{J}}$ from $\mathscr{H}$ into $\mathscr{K}_j$, let $\mathscr{H}^\Lambda$ be the set defined by

$$\mathscr{H}^\Lambda = \overline{\left\{\sum_{j\in\mathbb{I}} \Lambda_j^* g_j \text{ for any finite } \mathbb{I} \subset \mathbb{J} \text{ and } g_j \in \mathscr{K}_j,\ j \in \mathbb{I}\right\}}.$$

2 Preliminaries

In this section we mainly collect some basic definitions and properties for K-g-frames.

Definition 2.1 *Suppose $K \in B(\mathscr{H})$. One says that a family $\{\Lambda_j \in B(\mathscr{H}, \mathscr{K}_j)\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ if there exist $0 < C \le D < \infty$ such that*

$$C\|K^* f\|^2 \leqslant \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \leqslant D\|f\|^2, \qquad f \in \mathscr{H}. \tag{2.1}$$

The constants C and D are called, respectively, the lower and upper K-g-frame bounds.

Remark 2.1 If K is equal to the identity operator on $\mathscr{H}$, $\mathrm{Id}_{\mathscr{H}}$, then a K-g-frame turns to be a g-frame.

In general, if $\{\Lambda_j\}_{j\in\mathbb{J}}$ satisfies the inequality to the right in (2.1), we say that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a D-g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, associated with which there is a linear bounded operator, called the analysis operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$, defined by

$$U_\Lambda \colon \mathscr{H} \to \ell^2(\{\mathscr{K}_j\}_{j\in\mathbb{J}}), \qquad U_\Lambda f = \{\Lambda_j f\}_{j\in\mathbb{J}}. \tag{2.2}$$

The adjoint operator

$$U_\Lambda^* \colon \ell^2(\{\mathscr{K}_j\}_{j\in\mathbb{J}}) \to \mathscr{H}$$

given by

$$U_\Lambda^*\{g_j\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}} \Lambda_j^* g_j$$

is said to be the synthesis operator. A composition of U_Λ^* and U_Λ yields a positive and self-adjoint operator which we call the frame operator, given below

$$S_\Lambda\colon \mathscr{H} \to \mathscr{H}, \qquad S_\Lambda f = U_\Lambda^* U_\Lambda f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f. \tag{2.3}$$

It is easy to check that S_Λ is invertible if and only if $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a g-frame.

The following result tells us that we can naturally get a K-g-frame from a g-frame.

Proposition 2.1 *Suppose $K \in B(\mathscr{H})$. Then every g-frame is a K-g-frame.*

Proof. Let $\{\Lambda_j\}_{j\in\mathbb{J}}$ be a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with bounds C, D and the frame operator S_Λ. To prove that $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$, it only needs to show the lower bound inequality. For any $f \in \mathscr{H}$, we see from (2.3) that

$$f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j S_\Lambda^{-1} f$$

and thus,

$$Kf = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j S_\Lambda^{-1} Kf.$$

Now we compute that

$$\begin{aligned}
\|K^* f\|^2 &= \sup_{\|g\|=1} |\langle K^* f,\, g\rangle|^2 \\
&= \sup_{\|g\|=1} |\langle f,\, Kg\rangle|^2 \\
&= \sup_{\|g\|=1} \left| \sum_{j\in\mathbb{J}} \langle \Lambda_j^* \Lambda_j S_\Lambda^{-1} Kg,\, f\rangle \right|^2 \\
&= \sup_{\|g\|=1} \left| \sum_{j\in\mathbb{J}} \langle \Lambda_j S_\Lambda^{-1} Kg,\, \Lambda_j f\rangle \right|^2 \\
&\le \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \cdot \sup_{\|g\|=1} \sum_{j\in\mathbb{J}} \|\Lambda_j S_\Lambda^{-1} Kg\|^2 \\
&\le D\|S_\Lambda^{-1} K\|^2 \cdot \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2.
\end{aligned}$$

It follows that

$$D^{-1}\|S_\Lambda^{-1} K\|^{-2} \|K^* f\|^2 \leqslant \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2,$$

as desired.

Remark 2.2 The converse of Proposition 2.1 is not true, please check Example 3.1.

Proposition 2.2[22] *If $\Theta \in B(\mathscr{H}, \mathscr{K})$ has closed range, then there exists a unique operator $\Theta^\dagger \in B(\mathscr{K}, \mathscr{H})$ such that $\Theta\Theta^\dagger\Theta = \Theta$.*

Lemma 2.1[23] *Assume that $T\colon \mathscr{H} \to \mathscr{H}$ is a linear operator, and there are constants $\lambda, \mu \in [0, 1)$ satisfying*

$$\|Tf - f\| \leqslant \lambda\|f\| + \mu\|Tf\|, \qquad f \in \mathscr{H}.$$

Then $T \in B(\mathscr{H})$ and

$$\frac{1-\lambda}{1+\mu}\|f\| \leqslant \|Tf\| \leqslant \frac{1+\lambda}{1-\mu}\|f\|, \quad \frac{1-\mu}{1+\lambda}\|f\| \leqslant \|T^{-1}f\| \leqslant \frac{1+\mu}{1-\lambda}\|f\|, \qquad f \in \mathscr{H}.$$

3 The Main Result

The following result is presented in [21] as Theorem 4.1.

Theorem 3.1 *Suppose that $K \in B(\mathscr{H})$ has closed range and $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with bounds C and D. Suppose that $\Gamma_j \in B(\mathscr{H}, \mathscr{K}_j)$ for each $j \in \mathbb{J}$ and there exist constants $\lambda_1, \lambda_2, \mu \geqslant 0$ such that $\max\left\{\lambda_1 + \frac{\mu}{\sqrt{C}}, \lambda_2\right\} < 1$ and*

$$\left\|\sum_{j\in\mathbb{I}}(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j)f\right\| \leqslant \lambda_1\left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\| + \lambda_2\left\|\sum_{j\in\mathbb{I}}\Gamma_j^*\Gamma_j f\right\| + \mu\left(\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}$$

for any finite $\mathbb{I} \subset \mathbb{J}$ and any $f \in \mathscr{H}$, then $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with bounds

$$\frac{C\left[1 - \left(\lambda_1 + \frac{\mu}{\sqrt{C}}\right)\right]}{1+\lambda_2}, \qquad \frac{D\left(1 + \lambda_1 + \frac{\mu}{\sqrt{D}}\right)}{1-\lambda_2}.$$

In the proof of the lower K-g-frame bound condition the authors of [21] asserted that the frame operator of the associated K-g-frame is invertible on $\mathscr{H}$, implying that a K-g-frame is necessarily a g-frame, which is not true, as shown in the following example.

Example 3.1 Let $\{e_j\}_{j\in\mathbf{N}}$ be an orthonormal basis for $\mathscr{H}$ and define

$$K\colon \mathscr{H} \to \mathscr{H}, \qquad Kf = \sum_{j\in\mathbf{N}}\langle f, e_{2j}\rangle e_{2j}.$$

Clearly, K is well defined, linear bounded and self-adjoint. For $j = 1, 2, 3, \cdots$, let $\mathscr{K}_j = \overline{\operatorname{span}}\{e_j\}$ and define linear bounded operator $\Lambda_j\colon \mathscr{H} \to \mathscr{K}_j$ as follows:

$$\Lambda_j f = \begin{cases} \langle f, e_j\rangle e_j & \text{if } j \text{ is even;} \\ \langle f, e_j/j\rangle e_j & \text{if } j \text{ is odd.} \end{cases}$$

It is easily seen that the adjoint operator $\Lambda_j^*\colon \mathscr{K}_j \to \mathscr{H}$ is given by

$$\Lambda_j^* g_j = \begin{cases} \langle g_j, e_j\rangle e_j & \text{if } j \text{ is even;} \\ \langle g_j, e_j/j\rangle e_j & \text{if } j \text{ is odd.} \end{cases}$$

For any $f \in \mathscr{H}$ we have

$$\begin{aligned}\|K^*f\|^2 &= \sum_{j\in\mathbf{N}} |\langle f, e_{2j}\rangle|^2 \\ &\leqslant \sum_{j\in\mathbf{N}} \|\Lambda_j f\|^2 \\ &= \sum_{j\in\mathbf{N}} \|\Lambda_{2j} f\|^2 + \sum_{j\in\mathbf{N}} \|\Lambda_{2j-1} f\|^2 \\ &= \sum_{j\in\mathbf{N}} |\langle f, e_{2j}\rangle|^2 + \sum_{j\in\mathbf{N}} \frac{|\langle f, e_{2j-1}\rangle|^2}{(2j-1)^2} \\ &\leqslant \|f\|^2.\end{aligned}$$

Thus $\{\Lambda_j\}_{j\in\mathbf{N}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbf{N}}$.

We next show that $\{\Lambda_j\}_{j\in\mathbf{N}}$ does not satisfy the lower bound inequality of a g-frame. Assume on the contrary that there exists a constant $C > 0$ such that

$$C\|f\|^2 \leqslant \sum_{j\in\mathbf{N}} \|\Lambda_j f\|^2, \qquad f \in \mathscr{H}.$$

Let $k \in \mathbf{N}$ be a positive integer which is greater than $\dfrac{1}{2\sqrt{C}} + \dfrac{1}{2}$. Noting that $(2k-1)e_{2k-1} \in \mathscr{K}_{2k-1}$, we have

$$e_{2k-1} = \left\langle (2k-1)e_{2k-1}, \frac{e_{2k-1}}{2k-1} \right\rangle e_{2k-1} = \Lambda_{2k-1}^*((2k-1)e_{2k-1}) \in \mathscr{H}.$$

Therefore,

$$C = C\|e_{2k-1}\|^2 \leqslant \sum_{j\in\mathbf{N}} \|\Lambda_j e_{2k-1}\|^2 = \sum_{j\in\mathbf{N}} \left| \left\langle e_{2k-1}, \frac{e_{2j-1}}{2j-1} \right\rangle \right|^2 = \frac{1}{(2k-1)^2} < C,$$

a contradiction.

Suppose that $K \in B(\mathscr{H})$ and $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with bounds C, D and the frame operator S_Λ. From Example 3.1 we know that S_Λ is not an invertible operator on $\mathscr{H}$ in general. We show, however, that it can be invertible on the subspace $\mathscr{R}(K)$, provided K has closed range. Indeed, since $\mathscr{R}(K)$ is closed, from Proposition 2.2 it follows that

$$KK^\dagger f = f, \qquad f \in \mathscr{R}(K),$$

namely,

$$KK^\dagger|\mathscr{R}(K) = \mathrm{Id}_{\mathscr{R}(K)},$$

and $\mathrm{Id}^*_{\mathscr{R}(K)} = (K^\dagger|\mathscr{R}(K))^*K^*$ as a consequence. Thus

$$\|f\| = \|(K^\dagger|\mathscr{R}(K))^*K^*f\| \leqslant \|K^\dagger\|\,\|K^*f\|, \qquad f \in \mathscr{R}(K).$$

From this and taking into account that

$$\langle S_\Lambda f,\, f\rangle = \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2,$$

we obtain

$$D\|f\|^2 \geqslant \langle S_\Lambda f,\, f\rangle \geqslant C\|K^* f\|^2 \geqslant C\|K^\dagger\|^{-2}\|f\|^2, \qquad f \in \mathscr{R}(K),$$

implying that $S_\Lambda\colon \mathscr{R}(K) \to S_\Lambda(\mathscr{R}(K))$ is homeomorphous. Hence

$$D^{-1}\|f\| \leqslant \|S_\Lambda^{-1} f\| \leqslant C^{-1}\|K^\dagger\|^2\|f\|, \qquad f \in S_\Lambda(\mathscr{R}(K)).$$

Now we can improve Theorem 3.1 as follows.

Theorem 3.2 *Suppose that $K \in B(\mathscr{H})$ has closed range and $\{\Lambda_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Let C, D and S_Λ be the bounds and the frame operator of $\{\Lambda_j\}_{j\in\mathbb{J}}$, respectively. Suppose that $\Gamma_j \in B(\mathscr{H},\, \mathscr{K}_j)$ for each $j \in \mathbb{J}$ and that $\mathscr{R}(K)$, $\mathscr{H}^\Gamma \subset S_\Lambda(\mathscr{R}(K))$. If there exist constants λ_1, λ_2, $\mu \geq 0$ such that* $\max\left\{\lambda_1 + \dfrac{\mu\|K^\dagger\|}{\sqrt{C}},\ \lambda_2\right\} < 1$ *and*

$$\left\|\sum_{j\in\mathbb{I}}(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j)f\right\| \leqslant \lambda_1\left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\| + \lambda_2\left\|\sum_{j\in\mathbb{I}}\Gamma_j^*\Gamma_j f\right\| + \mu\left(\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}$$

for any finite $\mathbb{I} \subset \mathbb{J}$ and any $f \in \mathscr{H}$, then $\{\Gamma_j\}_{j\in\mathbb{J}}$ is a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$ with bounds

$$\frac{C^2\left[1 - \left(\lambda_1 + \dfrac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\right]^2(1-\lambda_2)}{D(1+\lambda_2)^2\|K^\dagger\|^4\|K\|^2\left(1 + \lambda_1 + \dfrac{\mu}{\sqrt{D}}\right)}, \qquad \frac{D\left(1 + \lambda_1 + \dfrac{\mu}{\sqrt{D}}\right)}{1-\lambda_2}.$$

Proof. For any finite $\mathbb{I} \subset \mathbb{J}$ and any $f \in \mathscr{H}$ we have

$$\begin{aligned}\left\|\sum_{j\in\mathbb{I}}\Gamma_j^*\Gamma_j f\right\| &\leqslant \left\|\sum_{j\in\mathbb{I}}(\Lambda_j^*\Lambda_j - \Gamma_j^*\Gamma_j)f\right\| + \left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\| \\ &\leqslant (1+\lambda_1)\left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\| + \lambda_2\left\|\sum_{j\in\mathbb{I}}\Gamma_j^*\Gamma_j f\right\| + \mu\left(\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

It follows that

$$\begin{aligned}\left\|\sum_{j\in\mathbb{I}}\Gamma_j^*\Gamma_j f\right\| &\leqslant \frac{1+\lambda_1}{1-\lambda_2}\left\|\sum_{j\in\mathbb{I}}\Lambda_j^*\Lambda_j f\right\| + \frac{\mu}{1-\lambda_2}\left(\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\ &\leqslant \frac{(1+\lambda_1)\sqrt{D}}{1-\lambda_2}\left(\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} + \frac{\mu}{1-\lambda_2}\left(\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\ &= \frac{(1+\lambda_1)\sqrt{D}+\mu}{1-\lambda_2}\left(\sum_{j\in\mathbb{I}}\|\Lambda_j f\|^2\right)^{\frac{1}{2}} \\ &\leqslant \frac{(1+\lambda_1)D + \mu\sqrt{D}}{1-\lambda_2}\|f\|,\end{aligned}$$

where in the second inequality we apply the fact that

$$\begin{aligned}\left\|\sum_{j\in\mathbb{I}}\varLambda_j^*\varLambda_j f\right\| &= \sup_{\|h\|=1}\left|\left\langle\sum_{j\in\mathbb{I}}\varLambda_j^*\varLambda_j f,\ h\right\rangle\right| \\ &= \sup_{\|h\|=1}\left|\sum_{j\in\mathbb{I}}\langle\varLambda_j f,\ \varLambda_j h\rangle\right| \\ &\leqslant \left(\sum_{j\in\mathbb{I}}\|\varLambda_j f\|^2\right)^{\frac{1}{2}}\sup_{\|h\|=1}\left(\sum_{j\in\mathbb{I}}\|\varLambda_j h\|^2\right)^{\frac{1}{2}} \\ &\leqslant \sqrt{D}\left(\sum_{j\in\mathbb{I}}\|\varLambda_j f\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Hence the series $\sum\limits_{j\in\mathbb{J}}\varGamma_j^*\varGamma_j f$ converges unconditionally in $\mathscr{H}$. Define

$$T\colon \mathscr{H}\to\mathscr{H},\qquad Tf=\sum_{j\in\mathbb{J}}\varGamma_j^*\varGamma_j f.$$

Then T is bounded with $\|T\|\leqslant\dfrac{D\left(1+\lambda_1+\frac{\mu}{\sqrt{D}}\right)}{1-\lambda_2}$. Thus for any $f\in\mathscr{H}$,

$$\sum_{j\in\mathbb{J}}\|\varGamma_j f\|^2=\langle Tf,\ f\rangle\leqslant\|T\|\,\|f\|^2\leqslant\frac{D\left(1+\lambda_1+\frac{\mu}{\sqrt{D}}\right)}{1-\lambda_2}\|f\|^2,$$

meaning that $\{\varGamma_j\}_{j\in\mathbb{J}}$ is a $\dfrac{D\left(1+\lambda_1+\frac{\mu}{\sqrt{D}}\right)}{1-\lambda_2}$-g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{K}_j\}_{j\in\mathbb{J}}$. Note that for any $f\in\mathscr{H}$ we also have

$$\left\|\sum_{j\in\mathbb{J}}(\varLambda_j^*\varLambda_j-\varGamma_j^*\varGamma_j)f\right\|\leqslant\lambda_1\left\|\sum_{j\in\mathbb{J}}\varLambda_j^*\varLambda_j f\right\|+\lambda_2\left\|\sum_{j\in\mathbb{J}}\varGamma_j^*\varGamma_j f\right\|+\mu\left(\sum_{j\in\mathbb{J}}\|\varLambda_j f\|^2\right)^{\frac{1}{2}}.$$

Then for each $g\in S_\varLambda(\mathscr{R}(K))$, taking $S_\varLambda^{-1}g$ instead of f in above equation we get

$$\begin{aligned}\|g-TS_\varLambda^{-1}g\| &\leqslant \lambda_1\|g\|+\lambda_2\|TS_\varLambda^{-1}g\|+\mu\left(\sum_{j\in\mathbb{J}}\|\varLambda_j S_\varLambda^{-1}g\|^2\right)^{\frac{1}{2}} \\ &\leqslant\left(\lambda_1+\frac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\|g\|+\lambda_2\|TS_\varLambda^{-1}g\|.\end{aligned}$$

Hence by Lemma 2.1,

$$L=TS_\varLambda^{-1}\colon S_\varLambda(\mathscr{R}(K))\to S_\varLambda(\mathscr{R}(K))$$

is invertible with

$$\|L^{-1}\|\leqslant\frac{1+\lambda_2}{1-\left(\lambda_1+\frac{\mu\|K^\dagger\|}{\sqrt{C}}\right)}.$$

Now for any $g \in S_\Lambda(\mathscr{R}(K))$,

$$\begin{aligned}\|g\|^2 &= \langle LL^{-1}g,\, g\rangle \\ &\leqslant \|L^{-1}\|\,\|g\|\,\|L^*g\| \\ &\leqslant \frac{1+\lambda_2}{1-\left(\lambda_1+\dfrac{\mu\|K^\dagger\|}{\sqrt{C}}\right)}\|S_\Lambda^{-1}\|\,\|g\|\,\|Tg\| \\ &\leqslant \frac{(1+\lambda_2)\|K^\dagger\|^2}{C\left[1-\left(\lambda_1+\dfrac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\right]}\frac{\sqrt{(1+\lambda_1)D+\mu\sqrt{D}}}{\sqrt{1-\lambda_2}}\|g\|\left(\sum_{j\in\mathbb{J}}\|\Gamma_j g\|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j\in\mathbb{J}}\|\Gamma_j g\|^2 &\geqslant \frac{C^2\left[1-\left(\lambda_1+\dfrac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\right]^2(1-\lambda_2)}{D(1+\lambda_2)^2\|K^\dagger\|^4\left(1+\lambda_1+\dfrac{\mu}{\sqrt{D}}\right)}\|g\|^2 \\ &\geqslant \frac{C^2\left[1-\left(\lambda_1+\dfrac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\right]^2(1-\lambda_2)}{D(1+\lambda_2)^2\|K^\dagger\|^4\|K\|^2\left(1+\lambda_1+\dfrac{\mu}{\sqrt{D}}\right)}\|K^*g\|^2.\end{aligned}$$

Since every $f \in \mathscr{H}$ has a composition as $f = g + h$, where $g \in S_\Lambda(\mathscr{R}(K))$ and $h \in (S_\Lambda(\mathscr{R}(K)))^\perp$, and $\sum_{j\in\mathbb{J}} \Gamma_j^*\Gamma_j f \in S_\Lambda(\mathscr{R}(K))$, it follows that

$$\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2 = \sum_{j\in\mathbb{J}}(\|\Gamma_j g\|^2+\|\Gamma_j h\|^2) + 2\mathrm{Re}\left\{\left\langle \sum_{j\in\mathbb{J}}\Gamma_j^*\Gamma_j g,\, h\right\rangle\right\} = \sum_{j\in\mathbb{J}}(\|\Gamma_j g\|^2+\|\Gamma_j h\|^2).$$

Noting also that $h \in (S_\Lambda(\mathscr{R}(K)))^\perp \subset (\mathscr{R}(K))^\perp$, we have

$$K^*f = K^*(g+h) = K^*g.$$

Thus

$$\begin{aligned}&\frac{C^2\left[1-\left(\lambda_1+\dfrac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\right]^2(1-\lambda_2)}{D(1+\lambda_2)^2\|K^\dagger\|^4\|K\|^2\left(1+\lambda_1+\dfrac{\mu}{\sqrt{D}}\right)}\|K^*f\|^2 \\ &= \frac{C^2\left[1-\left(\lambda_1+\dfrac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\right]^2(1-\lambda_2)}{D(1+\lambda_2)^2\|K^\dagger\|^4\|K\|^2\left(1+\lambda_1+\dfrac{\mu}{\sqrt{D}}\right)}\|K^*g\|^2 \\ &\leqslant \sum_{j\in\mathbb{J}}\|\Gamma_j g\|^2 \\ &\leqslant \sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2.\end{aligned}$$

Altogether we know that

$$\frac{C^2\left[1-\left(\lambda_1+\frac{\mu\|K^\dagger\|}{\sqrt{C}}\right)\right]^2(1-\lambda_2)}{D(1+\lambda_2)^2\|K^\dagger\|^4\|K\|^2\left(1+\lambda_1+\frac{\mu}{\sqrt{D}}\right)}\|K^*f\|^2 \leqslant \sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2 \leqslant \frac{D\left(1+\lambda_1+\frac{\mu}{\sqrt{D}}\right)}{1-\lambda_2}\|f\|^2$$

for every $f \in \mathscr{H}$, and the proof is finished.

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