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# The Existence of Weak Solutions of a Higher Order Nonlinear Eilliptic Equation

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**Abstract:** In this paper, we show the existence of weak solutions for a higher order nonlinear elliptic equation. Our main method is to show that the evolution operator satisfies the fixed point theorem for Banach semilattice.

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## 1 Introduction and Main Result

In this paper, we consider the following elliptic problem:

$$(-\Delta)^m u = \lambda V(x)u + h(u), \quad x \in \mathbf{R}^N,\ N \geq 3, \tag{$P_\lambda$}$$

where $\lambda$ is a positive parameter, $m \in \mathbf{N}$, $0 < m < \dfrac{N}{2}$. We make the assumptions over $V$ and $h$ as:

(V) $V(x) \in C(\mathbf{R}^N, \mathbf{R}^+)$, and there exists an $a \in \mathbf{R}$ such that

$$\lim \sup_{|x| \to \infty} V(x)|x|^a < \infty, \tag{1.1}$$

the index $a$ describes the property of $V$ near infinity, and we assume $a > 2m$.

(H) $h : \mathbf{R}^N \to \mathbf{R}$ is a measurable function, and the function $h(s)$ is nondecreasing on $\mathbf{R}$, there exists a positive constant $C$ such that $h$ holds the growth condition:

$$|h(s)| \leq C|s|^{\frac{N+2m}{N-2m}}, \quad s \in \mathbf{R}. \tag{1.2}$$

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Let $C_c^\infty(\mathbf{R}^N)$ denote the collection of smooth functions with compact support. Denote by $\mathcal{D}^{m,2}(\mathbf{R}^N)$ the completion of $C_c^\infty(\mathbf{R}^N)$ under the norm

$$\|u\| = \begin{cases} \left(\int_{\mathbf{R}^N} |\Delta^k u|^2 \mathrm{d}x\right)^{\frac{1}{2}} & \text{if } m = 2k; \\ \left(\int_{\mathbf{R}^N} |\nabla(\Delta^k u)|^2 \mathrm{d}x\right)^{\frac{1}{2}} & \text{if } m = 2k+1. \end{cases}$$

It is easy to see that the above norm comes from the scalar product

$$(u,\, v)_{\mathcal{D}^{m,\,2}(\mathbf{R}^N)} = \begin{cases} \int_{\mathbf{R}^N} (\Delta^k u)(\Delta^k v) \mathrm{d}x & \text{if } \ m = 2k; \\ \int_{\mathbf{R}^N} \nabla(\Delta^k u)\cdot\nabla(\Delta^k v) \mathrm{d}x & \text{if } \ m = 2k+1. \end{cases}$$

In this work, we consider our model problem in the working space $\mathcal{D}^{m,\,2}(\mathbf{R}^N)$.

The main result of the paper is the following theorem:

**Theorem 1.1** *If the assumptions $(V)$ and $(H)$ hold, then there exists a positive constant $\lambda_* > 0$ such that for any $0 < \lambda < \lambda_*$ the model problem $(P_\lambda)$ has a nontrivial weak solution.*

There are many methods to study nonlinear elliptic equations, such as the theory of monotone operators, the Schauder's fixed point theorem for compact mapping, the upper and lower solutions method, non-smooth critical point theories etc., we can see [1] for the choice of the method.

In this paper, we use the method introduced by Heikkilä[2] to show the existence of weak solutions of the elliptic problem, the main work is to show that the developed operator is increasing in order Banach spaces, then, due to the fixed point theorem, we get that the solution of the model problem is just the fixed point. The approach has been used in many work, one can see [3]–[9]. Here we give a solution in higher-order Sobolev space.

For the paper, the letter $C$ is a positive constant which maybe variety at different lines. We denote the norm of the Lebesgue space $L^p(\mathbf{R}^N)$ $(1 < p < \infty)$ as

$$\|\cdot\|_p = \left(\int_{\mathbf{R}^N} |\cdot|^p \mathrm{d}x\right)^{\frac{1}{p}}.$$

# 2 Preliminary

In this section, we give some preliminary results which are used later. Firstly, we give a special case of the result on Sobolev embedding which one can see [6].

**Lemma 2.1** *The space $\mathcal{D}^{m,\,2}(\mathbf{R}^N)$ is continuously embedding into the Lebesgue space $L^{\frac{2N}{N-2m}}(\mathbf{R}^N)$.*

Next, we give the definition of weak solution of the problem $(P_\lambda)$ in $\mathcal{D}^{m,2}(\mathbf{R}^N)$.

**Definition 2.1** *A function $u \in \mathcal{D}^{m,\,2}(\mathbf{R}^N)$ is a weak solution of the problem $(P_\lambda)$ if there holds*

$$\int_{\mathbf{R}^N} (-\Delta)^m uv \mathrm{d}x = \lambda \int_{\mathbf{R}^N} V(x)uv \mathrm{d}x + \int_{\mathbf{R}^N} h(u)u \mathrm{d}x.$$

*Due to the integral by part, the above equation implies*

$$(u,\, v)_{\mathcal{D}^{m,\,2}(\mathbf{R}^N)} = \lambda \int_{\mathbf{R}^N} V(x)uv\mathrm{d}x + \int_{\mathbf{R}^N} h(u)u\mathrm{d}x.$$

Now we give the concept of Banach semilattice, one can see [6]. Let $X$ be a real Banach space. A nonempty set $X_+$ of $X$ is said to be an order cone if the following conditions hold:

(i) $X_+$ is a convex and closed set;

(ii) for any $u \in X_+$ and a real number $\beta \geq 0$, $\beta u \in X_+$;

(iii) if $u \in X_+$ and $-u \in X_+$, then $u = 0$.

A partial order in $X$ derives from the order cone $X_+$ as: $x \prec y$ if and only if $(y-x) \in X_+$, and $(X, \prec)$ is called an ordered Banach space. If for any $x, y \in X$, then $\inf\{x,\, y\}$ and $\sup\{x,\, y\}$ exist with respect to $\prec$, then we call $X$ to be a lattice. Moreover, if for any $x \in X$, we set $x^+ = \sup\{x,\, 0\}$ and $x^- = \inf\{x,\, 0\}$, the inequality $\|x^\pm\| \leq \|x\|$ holds, then we call $(X,\, \|\cdot\|)$ to be a Banach semilattice.

Now suppose that $(X,\, \prec)$ and $(Y,\, \lhd)$ are two ordered Banach spaces. We call an operator $F : X \to Y$ to be increasing if and only if for all $x,\, y \in X$, $x \prec y$ implies $F(x) \lhd F(y)$. A subset $B$ of $X$ is said to have fixed point property if every increasing operator $\Lambda : B \to B$ has a fixed point.

Finally, we give the fixed point theorem due to Carl and Heikkilä[3]. We use this fixed point theorem to prove Theorem 1.1.

**Theorem 2.1** [3] *Set $X$ be a Banach semilattice which is also reflexive. Then any closed ball of $X$ has the fixed point property.*

# 3 Proof of Theorem 1.1

In this section, we show the proof of Theorem 1.1. First we give a operator, and prove the operator is a contract mapping. For any $\lambda > 0$, we define the operator $T_\lambda : \mathcal{D}^{m,2}(\mathbf{R}^N) \to (\mathcal{D}^{m,2}(\mathbf{R}^N))^*$ as

$$\langle T_\lambda u,\, v\rangle = \lambda \int_{\mathbf{R}^N} V(x)uv\mathrm{d}x + \int_{\mathbf{R}^N} h(u)v\mathrm{d}x, \quad v \in \mathcal{D}^{m,\,2}(\mathbf{R}^N),$$

where $(\mathcal{D}^{m,2}(\mathbf{R}^N))^*$ denotes the topological dual of $\mathcal{D}^{m,2}(\mathbf{R}^N)$. According to Riesz's representation theorem, for $u \in \mathcal{D}^{m,2}(\mathbf{R}^N)$ and $\lambda > 0$, there exists a unique element $L_\lambda u \in \mathcal{D}^{m,2}(\mathbf{R}^N)$ such that

$$(L_\lambda u,\, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} = \langle T_\lambda u, v\rangle, \quad v \in \mathcal{D}^{m,\,2}(\mathbf{R}^N).$$

Now we show that the operator $L_\lambda$ is a congruent mapping under some conditions.

**Lemma 3.1** *Suppose that $(V)$ and $(H)$ hold. There exist two positive constants $\lambda_*$ and $R$ such that for all $0 < \lambda < \lambda_*$,*

$$L_\lambda(B'(0,\, R)) \subset B'(0,\, R)$$

*holds, where*

$$B'(0,\, R) = \{u \in \mathcal{D}^{m,\,2}(\mathbf{R}^N),\, \|u\| \leq R\}.$$

*Proof.* Set $u \in \mathcal{D}^{m,2}(\mathbf{R}^N)$ and $\lambda > 0$. Hölder's inequality implies

$$\begin{aligned}(L_\lambda u,\, L_\lambda u)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} &= \lambda \int_{\mathbf{R}^N} V(x) u L_\lambda u \mathrm{d}x + \int_{\mathbf{R}^N} h(u) L_\lambda u \mathrm{d}x \\ &\le C\lambda \|V\|_{\frac{N}{2m}} \|u\|_{\frac{2N}{N-2m}} \|L_\lambda u\|_{\frac{2N}{N-2m}} + C\|h(u)\|_{\frac{2N}{N+2m}} \|L_\lambda u\|_{\frac{2N}{N-2m}}.\end{aligned}$$

Now, due to the assumption (1.1), we know that if $|x| > M$, where $M > 0$ is large enough, then $\sup V(x)|x|^a \sim C$, thus $V(x) \le C|x|^{-a}$, consequently,

$$\begin{aligned}\|V\|_{\frac{N}{2m}} &\le C\Big[\Big(\int_{|x|\le M} |V(x)|^{\frac{N}{2m}} \mathrm{d}x\Big)^{\frac{2m}{N}} + \Big(\int_{|x|>M} |V(x)|^{\frac{N}{2m}} \mathrm{d}x\Big)^{\frac{2m}{N}}\Big] \\ &\le C + C\Big(\int_{|x|>M} \frac{1}{|x|^{\frac{aN}{2m}}} \mathrm{d}x\Big)^{\frac{2m}{N}} \\ &\le C, \end{aligned} \tag{3.1}$$

where we used the continuity of the function $V$ on the bounded domain $\{x \in \mathbf{R}^N : |x| \le M\}$, and $a > 2m$. Next, by the condition (1.2), we have

$$\begin{aligned}\|h(u)\|_{\frac{2N}{N+2m}} &= \Big(\int_{\mathbf{R}^N} |h(u)|^{\frac{2N}{N+2m}} \mathrm{d}x\Big)^{\frac{N+2m}{2N}} \\ &\le C\Big(\int_{\mathbf{R}^N} |u|^{\frac{2N}{N-2m}} \mathrm{d}x\Big)^{\frac{N+2m}{2N}} \\ &= C\|u\|_{\frac{2N}{N-2m}}^{\frac{N+2m}{N-2m}}. \end{aligned} \tag{3.2}$$

Combining (3.1), (3.2) and the embedding in Lemma 2.1

$$\mathcal{D}^{m,2}(\mathbf{R}^N) \hookrightarrow L^{\frac{2N}{N-2m}}(\mathbf{R}^N),$$

we obtain

$$(L_\lambda u,\, L_\lambda u)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \le C_1(\lambda\|u\| + \|u\|^{\frac{N+2m}{N-2m}})\|L_\lambda u\|,$$

where $C_1$ is a constant. Then we have

$$\|L_\lambda u\| \le C_1(\lambda\|u\| + \|u\|^{\frac{N+2m}{N-2m}}).$$

Now we choose $0 < R < 1$ small enough, so that $C_1 R^{\frac{N+2m}{N-2m}} \le \dfrac{R}{2}$. We just choose $\lambda_* = \dfrac{1}{2C_1}$. Then for all $0 < \lambda < \lambda_*$,

$$\|L_\lambda u\| \le C_1(\lambda R + R^{\frac{N+2m}{N-2m}}) \le C_1 \lambda R + \frac{R}{2} < 1, \quad \|u\| \le R.$$

Therefore we complete the proof of the lemma.

Now we show the proof of Theorem 1.1. We bring in the positive cone

$$\mathcal{K}_+ = \{u \in \mathcal{D}^{m,2}(\mathbf{R}^N),\ u \ge 0,\ \text{almost every in } \mathbf{R}^N\}$$

to a partial order on $\mathcal{D}^{m,2}(\mathbf{R}^N)$ which is preserved by the operator $L_\lambda$. As in [6], we introduce another convex closed cone

$$\mathcal{K} = \{u \in \mathcal{D}^{m,2}(\mathbf{R}^N),\, (u,\, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \ge 0 \text{ for any } v \in \mathcal{K}_+\}.$$

In [10], the dual cone of $\mathcal{K}_+$ is

$$\mathcal{K}_+^* = \{u \in \mathcal{D}^{m,2}(\mathbf{R}^N),\ (u,\, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \le 0 \text{ for any } v \in \mathcal{K}_+\}.$$

By the definition of the negative cone

$$(-\mathcal{K}_+) = \mathcal{K}_- = \{u \in \mathcal{D}^{m,2}(\mathbf{R}^N),\ (u,\, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \le 0,\ \text{almost every in } \mathbf{R}^N\},$$

it is easy to see that

$$\mathcal{K}_+^* \subset \mathcal{K}_-. \tag{3.3}$$

According to [6], we know that $\mathcal{K}$ is an order cone, and $\mathcal{K}$ induces a partial order in $\mathcal{D}^{m,2}(\mathbf{R}^N)$ as

$$u_1 \prec u_2 \Leftrightarrow (u_2, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \geq (u_1, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)}, \quad v \in \mathcal{K}_+. \tag{3.4}$$

It is also known that $\mathcal{K}_+$ is a convex closed cone of the Hilbert space $\mathcal{D}^{m,2}(\mathbf{R}^N)$ and $\mathcal{D}^{m,2}(\mathbf{R}^N)$ is a Banach semilattice.

Next we show that for any $\lambda > 0$, the operator $L_\lambda$ introduced above is increasing from $(\mathcal{D}^{m,2}(\mathbf{R}^N), \prec)$ to $(\mathcal{D}^{m,2}(\mathbf{R}^N), \prec)$. Suppose that $u_1, u_2 \in \mathcal{D}^{m,2}(\mathbf{R}^N)$ and they satisfy $u_1 \prec u_2$. Thus by (3.5), we have $(u_2 - u_1, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \geq 0$ for any $v \in \mathcal{K}_+$, which implies $(u_1 - u_2, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \leq 0$ for any $v \in \mathcal{K}_+$. By the relation (3.4), we have $u_1 - u_2 \in \mathcal{K}_-^* \subset \mathcal{K}_-$, which means $u_1 \leq u_2$, a.e., $x \in \mathbf{R}^N$. Hence, due to $V \geq 0$, $\lambda > 0$ and the assumption (H) over the function $h$, we obtain that

$$\begin{aligned}
\langle T_\lambda u_1, v\rangle &= \lambda \int_{\mathbf{R}^N} V(x)u_1(x)v(x)\mathrm{d}x + \int_{\mathbf{R}^N} h(u_1)v(x)\mathrm{d}x \\
&\leq \lambda \int_{\mathbf{R}^N} V(x)u_2(x)v(x)\mathrm{d}x + \int_{\mathbf{R}^N} h(u_2)v(x)\mathrm{d}x \\
&= \langle T_\lambda u_2, v\rangle, \quad v \in \mathcal{K}_+.
\end{aligned}$$

That means that for any $v \in \mathcal{K}_+$, $(L_\lambda u_1, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)} \leq (L_\lambda u_2, v)_{\mathcal{D}^{m,2}(\mathbf{R}^N)}$, thus $L_\lambda u_1 \prec L_\lambda u_2$. Taking into Theorem 2.1 and Lemma 3.1 into account, we have that for any $0 < \lambda < \lambda_*$, the operator $L_\lambda$ has a fixed point, which is a weak solution of the problem $(P_\lambda)$. And the weak solution is nontrivial due to the fact $h \neq 0$. Therefore we complete the proof of Theorem 1.1.

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