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Expanders, Group Extensions, Hadamard Manifolds and Certain Banach Spaces

Shan Lin
(*Faculty of Natural Sciences, University of Puerto Rico, Rio Piedras Campus, San Juan, 00925, Puerto Rico*)

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Abstract: In this note, we prove that expanders cannot be coarsely embedded into group extensions of sequences of groups which are coarsely embeddable into Hardamad manifolds and certain Banach spaces due to the similar concentration theorems.

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In [1], Arzhantseva and Tessera proved the property that expanders cannot be coarsely embedded into group extensions of sequences of groups which are coarsely embeddable into Hilbert spaces. In this note, we show that this property also holds for Hadamard manifolds and Banach spaces whose unit balls are uniformaly embeddable into Hilbert spaces.

First, let us recall basic definitions of expanders (see [2] for more information and for references). Let (V, E) be a finite graph with the vertex set V and the edge set E. Denote the cardinality of V and E by $|V| = n$ and $|E| = m$. We also define an orientation on E. The differential $d : \ell_2(V) \to \ell_2(E)$ is defined by

$$d(f) = f(e^+) - f(e^-)$$

for all $f \in \ell_2(V)$ and $e = (e^+, e^-) \in E$, where e^+ and e^- are initial and end points of e, respectively.

This differential $\boldsymbol{d}$ is an $m \times n$ matrix. The discrete Laplace operator $\Delta = d^*\boldsymbol{d}$, where d^* is the adjoint operator of d. This definition does not depend on the choice of the orientation of E. Δ is self-adjoint and positive. Hence it has real nonnegative eigenvalues. Denote by $\lambda_1(V)$ the minimal positive eigenvalue of the discrete Laplace operator Δ of the graph (V, E).

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E-mail address: lin.shan@upr.edu.

Definition 1 *A (k, λ) expander is a graph (V, E) with a fixed degree k and $\lambda_1(V) \geq \lambda$. A sequence of graphs (V_n, E_n) of a fixed degree k and with $|V_n| \to \infty$ is called an expanding sequence of graphs if there is a positive constant λ such that $\lambda_1(V_n) \geq \lambda$ for all $n \in \mathbf{N}$.*

Let X and Y be metric spaces and let $f : X \to Y$ be a map. Define the Lipschitz constant of f by

$$\mathrm{Lip}(f) = \sup\left\{\frac{d(f(s),\, f(t))}{d(s,\, t)} : s,\, t \in X \text{ and } s \neq t\right\}.$$

Lemma 1 *If X is a graph, then*

$$Lip(f) = \sup\{d(f(s),\, f(t)) : s,\, t \in X \text{ are adjacent}\}.$$

Proof. Let

$$\mathrm{Lip}_1(f) = \sup\{d(f(s),\, f(t)) : s,\, t \in X \text{ are adjacent}\}.$$

Clearly $\mathrm{Lip}_1(f) \leq \mathrm{Lip}(f)$. For any pair $s,\, t \in X$ with $d(s,\, t) = n$, there exists a sequence of points $x_0,\, x_1,\, \cdots,\, x_n$ of X such that $x_0 = s$, $x_n = t$ and $x_i,\, x_{i+1}$ are adjacent for all $i = 0, \cdots, n-1$. Then

$$\begin{aligned}\frac{d(f(s),\, f(t))}{d(s,\, t)} &= \frac{d(f(s),\, f(t))}{n}\\ &\leq \sum_{i=0}^{n-1}\frac{d(f(x_i),\, f(x_{i+1}))}{n}\\ &\leq n\cdot\frac{\mathrm{Lip}_1(f)}{n}\\ &= \mathrm{Lip}_1(f).\end{aligned}$$

Therefore, $\mathrm{Lip}(f) \leq \mathrm{Lip}_1(f)$. The proof is done.

Hence, for any pair $(s,\, t) \in G$, one always has

$$d(f(s),\, f(t)) \leq \mathrm{Lip}(f)d(s,\, t).$$

In [4], we have the following concentration theorem.

Theorem 1[4] *Let M be a Hadamard manifold with bounded sectional curvatures. And let $(V,\, E)$ be a (k, λ) expander. Then there exists $R > 0$ such that for any $f : V \to M$,*

$$\frac{1}{|V|}\sum_{v\in V} d(f(v),\, m)^2 \leq R^2(\mathrm{Lip}(f) + 1)^2,$$

where m is a point such that $\sum\limits_{v\in V}\log_m(f(v)) = 0$.

In the following proposition, we see that if the average value is bounded by a number, then at least half of summands are bounded by the twice of the preceding number.

Proposition 1 *Let $a_1,\, \cdots,\, a_n$ be non-negative real numbers and $a > 0$. If*

$$\frac{1}{n}\sum_{i=1}^{n} a_i^2 \leq a^2,$$

then there are at least $\dfrac{n}{2}$ of $a_1,\, \cdots,\, a_n$ less than or equal to $2a$.

Proof. Assume that this is false. Then there are at least k numbers $a_{i_1}, \cdots, a_{i_k}$ such that $a_{i_j} > 2a$ and $k \geq \frac{n}{2}$. Then

$$\frac{1}{n}\sum_{i=1}^{n} a_i^2 \geq \frac{1}{n}\sum_{j=1}^{k} a_{i_j}^2 > \frac{1}{n}4a^2k > \frac{1}{n}4a^2\frac{n}{2} > 2a^2,$$

which contradicts the hypothesis of the proposition

$$\frac{1}{n}\sum_{i=1}^{n} a_i^2 \leq a^2.$$

Hence at least $\frac{n}{2}$ numbers of $a_1, \cdots, a_n$ are less than or equal to $2a$. The proof is done.

Corollary 1 *Under the assumption of the previous theorem, there is a subset A of G and a point $g \in G$ such that $|A| \geq \frac{1}{2}|G|$ and the points of A are mapped at distance at most $4R \cdot (\mathrm{Lip}(f) + 1)$ from $f(g)$.*

Proof. By the previous theorem and the proposition, choosing $a_v = \|f(v) - m\|$ for $v \in G$ and $a = R \cdot (\mathrm{Lip}(f) + 1)$, there is a subset A of G such that

$$\|f(s) - m\| \leq 2R \cdot (\mathrm{Lip}(f) + 1),$$

for all $s \in A$ and $|A| \geq \frac{1}{2}|G|$. Then for any points $s, g \in A$,

$$\|f(s) - f(g)\| \leq 4R \cdot (\mathrm{Lip}(f) + 1).$$

The proof is done.

Based on the concentration theorem and corollary, we now prove our main theorem.

Theorem 2 *Let Z be a Hadamard manifold with bounded sectional curvatures, and let $(G_n)_{n\in\mathbf{N}}$ be a sequence of finitely generated groups equipped with finite generating sets S_n of size k. We assume that for every n, there is an exact sequence*

$$1 \to N_n \to G_n \to Q_n \to 1$$

such that

(1) *the sequence $(N_n)_{n\in\mathbf{N}}$ equipped with the induced metric coarsely embeds into Z;*

(2) *the sequence $(Q_n)_{n\in\mathbf{N}}$ equipped with the word metric associated to the projection T_n of S_n coarsely embeds into Z.*

Then, given a number $K > 0$, an expander $(X_n)_{n\in\mathbf{N}}$, and a sequence of K-Lipschitz maps $h_n : X_n \to Y_n = (G_n, S_n)$, there exists a constant $c > 0$ and a sequence $y_n \in Y_n$ such that the cardinality of $h_n^{-1}(\{y_n\})$ is at least $c|X_n|$, and its diameter is not less than $c\,\mathrm{diam}(X_n)$.

Proof. By the hypothesis, there are two maps $\phi_n : (N_n, d_{S_n}) \to Z$ and $\psi_n : (Q_n, d_{T_n}) \to Z$ and two proper functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ with the property that for all $n \in \mathbf{N}$ and all $x, y \in N_n$, $\lim_{t\to\infty} \rho_-(t) = \infty$,

$$\rho_-(d_{S_n}(x, y)) \leq d(\phi_n(x), \phi_n(y)) \leq \rho_+(d_{S_n}(x, y)),$$

and

$$\rho_-(d_{T_n}(x, y)) \leq d(\psi_n(x), \psi_n(y)) \leq \rho_+(d_{T_n}(x, y)).$$

Let $\lambda > 0$, d be such that for all n, X_n is a (d, λ)-expander.

Assume that $h_n : X_n \to G_n$ are K-Lipschitz maps. Let π_n denote the projection map from G_n to Q_n for $n \in \mathbf{N}$. Clearly π_n is always 1-Lipschitz. We define the composition $f_n := \psi_n \circ \pi_n \circ h_n : X_n \to Z$. The metric space Q_n is equipped with the graph metric, so $\psi_n : Q_n \to Z$ has the Lipschitz constant

$$\mathrm{Lip}(\psi_n) = \sup\{d(\psi_n(s), \psi_n(t)), \text{ where } s, t \in Q_n \text{ are adjacent}\}.$$

Since

$$d(\psi_n(x), \psi_n(y)) \leq \rho_+(d_{T_n}(x, y))$$

for $x, y \in Q_n$ and $d_{T_n}(s, t) = 1$ if s, t are adjacent in Q_n,

$$\mathrm{Lip}(\psi_n) \leq \rho_+(1), \qquad n \in \mathbf{N}.$$

By the composition, $\{f_n : X_n \to Z\}_{n\in\mathbf{N}}$ is a sequence of $K\rho_+(1)$-Lipschitz maps. Applying Corollary 1 for each X_n, there exist $x_n \in X_n$ and a subset $A_n \subseteq X_n$ with $|A_n| \geq \dfrac{|X_n|}{2}$ such that $f_n(A_n) \subset B(f_n(x_n), 4R(K\rho_+(1) + 1))$. By the properness of ρ_-, it implies that $\pi_n \circ h_n(A_n) \subset B(\pi_n \circ h_n(x_n), r)$, where $r = \rho_-^{-1}(4R(K\rho_+(1) + 1))$ is independent of n. As in the hypothesis, G_n has the generating set S_n of size k and Q_n is equipped with the word metric associated to the projection T_n of S_n, the cardinality $|B(\pi_n \circ h_n(x_n), r)| \leq k^r$. Hence there is a subset A'_n of A_n of with $|A'_n| \geq \dfrac{|A_n|}{k^r} \geq \dfrac{|X_n|}{2k^r}$ whose image $\pi_n \circ h_n(A'_n)$ is one point in Q_n. Therefore, $h_n(A'_n)$ belongs to a coset of N_n in G_n. Composing h_n by a proper left translation in G_n, without loss of generality we may assume that $h_n(A'_n) \subset N_n$.

Next we repeat the preceding process. Consider the composition

$$g_n = \phi_n \circ h_n : A'_n \to Z.$$

With the similar estimation, $\mathrm{Lip}(g_n) \leq K\rho_+(1)$. Applying Corollary 4 to the map g_n, there exist sequences $y_n \in A'_n$ and $A''_n \subseteq A'_n$ with $|A''_n| \geq \dfrac{|A'_n|}{2}$, such that

$$g_n(A''_n) \subseteq B(g_n(y_n), 4R(K\rho_+(1) + 1)).$$

We conclude that, by the properness of ρ_-, $h_n(A''_n) \subseteq B(h_n(y_n), r)$, where

$$r = \rho_-^{-1}(4R(K\rho_+(1) + 1)).$$

Hence, there exists a subset A'''_n of A''_n with

$$|A'''_n| \geq \frac{|A''_n|}{k^r} \geq \frac{|A'_n|}{2k^r} \geq \frac{|X_n|}{4k^{2r}},$$

which is mapped by h_n to one point. This proves our theorem.

In [3], Ozawa showed the following concentration theorem.

Theorem 3 [3] *Let X ba a Banach space whose unit ball is uniformly embeddable into a Hilbert space. Then, for any $k \in \mathbf{N}$ and $h > 0$, there exists a positive number $R = R(k, h, X)$ which satisfies the following: for any map f from a (k, h)-expander G into X, we have*

$$\frac{1}{|G|}\sum_{s\in G}\|f(s) - m\| \leq R \cdot \mathrm{Lip}(f),$$

where $m = \dfrac{1}{|G|}\sum_{s\in G} f(s)$ is the mean of f.

With the similar proof, we can have the result for Banach spaces.

Theorem 4 *Let Z ba a Banach space whose unit ball is uniformly embeddable into a Hilbert space, and let $(G_n)_{n\in\mathbf{N}}$ be a sequence of finitely generated groups equipped with finite generating sets S_n of size k. We assume that for every n, there is an exact sequence*

$$1 \to N_n \to G_n \to Q_n \to 1$$

such that

(1) *the sequence $(N_n)_{n\in\mathbf{N}}$ equipped with the induced metric coarsely embeds into Z;*

(2) *the sequence $(Q_n)_{n\in\mathbf{N}}$ equipped with the word metric associated to the projection T_n of S_n coarsely embeds into Z.*

Then, given a number $K > 0$, an expander $(X_n)_{n\in\mathbf{N}}$, and a sequence of K-Lipschitz maps $h_n : X_n \to Y_n = (G_n, S_n)$, there exists a constant $c > 0$ and a sequence $y_n \in Y_n$ such that the cardinality of $h_n^{-1}(\{y_n\})$ is at least $c|X_n|$, and its diameter is not less than $c\,\mathrm{diam}(X_n)$.

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