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Global Existence and Blow-up for a Two-dimensional Attraction-repulsion Chemotaxis System

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Abstract: This paper is devoted to dealing with the parabolic-elliptic-elliptic attraction-repulsion chemotaxis system. We aim to understand the competition among the repulsion, the attraction, the nonlinear productions and give conditions of global existence and blow-up for the two-dimensional attraction-repulsion chemotaxis system.
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1 Introduction

Chemotaxis is a phenomenon which describes the movement of cells in response to the concentration gradient of the chemical produced by cells themselves. The famous chemotaxis model was first proposed by Keller and Segel[1] in 1970. The Keller-Segel model can be read as follows:

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(\chi u\nabla v) + f(u), & x\in\Omega,\ t\in(0,\,T),\\ \tau v_t = \Delta v + u - v, & x\in\Omega,\ t\in(0,\,T),\end{cases} \tag{1.1}$$

where $u = u(x,\,t)$ denotes the density of cells, $v = v(x,\,t)$ represents the concentration of the chemoattractant. The function $f : [0,\,\infty) \longrightarrow \mathbf{R}$ is smooth, χ is the parameter referred as chemosensitivity. The system (1.1) with $\tau = 0$ or $\tau = 1$ has been studied extensively in the past four decades. For instance, when $D(u) \equiv 1$, $\tau = 0$ and $\Omega \subset \mathbf{R}^2$ is a bounded

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domain, the solution to (1.1) is global bounded provided that $f(u) = \mu u(1-u)$ with $\mu > 0$ (see [2]).

In order to better understand the parabolic-elliptic-elliptic attraction-repulsion chemotaxis system, let us mention previous contributions as follows:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0. \end{cases} \tag{1.2}$$

When $n = 1$, the main results in [3] showed that the (1.2) admits a unique global solution. Nagai[4] found that there exists the critical mass $m_c = \dfrac{8\pi}{\chi\alpha}$ which determines the behavior of the solution when $n = 2$. Precisely, if the initial mass $\int_\Omega u_0(x)\mathrm{d}x \le m_c$, the solution of system (1.2) is global and bounded, whereas the finite time blow-up happens when $\int_\Omega u_0(x)\mathrm{d}x > m_c$. In addition, the blow-up may occur when $\int_\Omega u_0(x)\mathrm{d}x > \dfrac{4\pi}{\chi\alpha}$ in some special Ω (see [5]–[8]). When $n \ge 3$, Winker[9] showed that there exists radially symmetric solution blowing up in finite time with proper initial conditions.

In numerous biological processes, general mechanisms in cell include not only attractive but also repulsive signals, which can form various interesting biological patterns (see [10]). Then the model can be expressed as following attraction-repulsion chemotaxis system:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t\in(0,T),\\ \tau v_t = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t\in(0,T),\\ \tau w_t = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t\in(0,T). \end{cases} \tag{1.3}$$

The system (1.3) is proposed by [11] to describe the aggregation of microglia observed in Alzheimer's disease. Fewer blow-up results are available for system (1.3) than (1.1), because (1.3) relies on Lyapunov function. When $n = 1$, $\tau = 1$, global existence, non-trivial stationary, asymptotic behavior and pattern formation of solutions to the system (1.3) have been studied in [12]–[13]; when $n = 2$, $\tau = 1$, if $\beta \ne \delta$ and repulsion prevails over attraction (i.e., $\xi\gamma - \chi\alpha > 0$), then the system (1.3) admits a unique global bounded solution (see [14]); in the case $\tau \equiv 0$, Yu *et al.*[15] proved that the finite time blow-up for the nonradial solution happens when

$$\chi\alpha - \xi\gamma > 0, \qquad \int_\Omega u_0(x)\mathrm{d}x > \frac{8\pi}{\chi\alpha - \xi\gamma}$$

and $\int_\Omega u_0(x)|x - x_0|^2\mathrm{d}x$ sufficiently small for some $x_0 \in \Omega$; Li and Li[16] showed that the finite time blow-up for the nonradial solution of (1.3) happens under either the condition

$$\chi\alpha - \xi\gamma > 0, \qquad \int_\Omega u_0(x)\mathrm{d}x > \frac{8\pi}{\chi\alpha - \xi\gamma}$$

when $\delta \ge \beta$, or

$$\chi\alpha\delta - \xi\gamma\beta > 0, \qquad \int_\Omega u_0(x)\mathrm{d}x > \frac{8\pi}{\chi\alpha\delta - \xi\gamma\beta}$$

if $\delta < \beta$. For more detail results on attraction-repulsion chemotaxis system, we refer the readers to [17]–[20]. Blow-up is an extremely behavior. In order to restrain the behavior, people add the logistic source. More detail results on attraction-repulsion chemotaxis system with the logistic source, the readers can see [21]–[24].

Inspired by above papers, this paper mainly aims to understand the competition among the repulsion, the attraction, the nonlinear productions. Precisely, we consider the global boundedness and the finite time blow-up of solutions to the following parabolic-elliptic-elliptic attraction-repulsion chemotaxis system:

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w), & x\in\Omega,\ t\in(0,\,T), \\ 0 = \Delta v + \alpha u^q - \beta v, & x\in\Omega,\ t\in(0,\,T), \\ 0 = \Delta w + \gamma u^r - \delta w, & x\in\Omega,\ t\in(0,\,T), \\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t\in(0,\,T), \\ u(x,\,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where $u = u(x,\,t)$ denotes the density of cells, $v = v(x,\,t)$ represents the concentration of the chemoattractant, $w = w(x,\,t)$ is a secondary chemical signal as a chemorepellent which mediates the cell's chemotactic response to the chemoattractant v. Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$, $\dfrac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal. $\chi,\,\xi>0$, $\alpha,\,\beta,\,\gamma,\,\delta>0$, $q\geq 1$, $r\geq 1$ and the nonnegative initial data $u_0(x)\in C^0(\bar{\Omega})$. In this model, the behavior of solutions relies on the interaction between the attraction and repulsion, with nonlinear productions.

The main results of this paper read as follows:

Theorem 1.1 (i) *If $q<r$, then the system* (1.4) *has a unique nonnegative and globally bounded solution*;

(ii) *If $q=r$ and $\chi\alpha-\xi\gamma<0$, then the solution of* (1.4) *is globally bounded.*

Theorem 1.2 *Let $\Omega\in\mathbf{R}^2$ be a smooth bounded domain and $x_0\in\Omega$. Assume that $\displaystyle\int_\Omega u_0(x)|x-x_0|^2\mathrm{d}x$ is small enough. Then if either of the following case holds*:

(i) $q=r$, $\chi\alpha-\xi\gamma>0$, *and* $\displaystyle\left(\int_\Omega u_0(x)\mathrm{d}x\right)^r > \frac{8\pi|\Omega|^{r-1}}{\chi\alpha-\xi\gamma}$;

(ii) $q>r$, $\chi\alpha q-\xi\gamma r>0$, $\displaystyle\left(\int_\Omega u_0(x)\mathrm{d}x\right)^q > \frac{2\pi q|\Omega|^{q-1}}{\chi\alpha q-\xi\gamma r}\left(4+\frac{\xi\gamma(q-r)|\Omega|}{2\pi q}\right)$,

the solution of (1.4) *blows up in finite time.*

This paper is structured as follows. In Section 2, we collect some preliminaries which are used later. Section 3 is devoted to proving Theorem 1.1. Finally, we give some blow-up conditions for the system (1.4), and prove Theorem 1.2 in Section 4.

2 Preliminaries

In this section, we collect some classical conclusions as preliminaries. We begin with the local existence of solutions to (1.4).

Lemma 2.1 [7],[15],[16] *Assume that $u_0(x)\in C^0(\bar{\Omega})$ is nonnegative in Ω. Then there exists a unique triple $(u,\,v,\,w)$ of nonnegative functions from $C^0(\Omega\times(0,\,T_{\max}))\bigcap C^{2,\,1}(\Omega\times$*

$(0, T_{\max}))$ *with* $T_{\max} \in (0, \infty]$ *solving* (1.4) *in the classical sense. Furthermore, if* $T_{\max} < \infty$, *then*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \qquad as \quad t \nearrow T_{\max}.$$

Lemma 2.2 [7],[15],[16] *Let u be the solution of*

$$\begin{cases} -\Delta u = f, & x \in B, \\ u = 0, & x \in \partial B, \end{cases}$$

where $B := \{x \in \mathbf{R}^2 \mid |x| < R\}$ *and* $f \in L^p(B)$, $1 \le p \le \infty$. *Then*

$$u(x) = \int_B G(x, y) f(y) \mathrm{d}y, \qquad x \in B,$$

where $G(x, y)$ is the green function of $-\Delta$ on B with homogeneous Dirichlet boundary condition. Besides, $G(x, y)$ satisfies the following properties:

(i) $G(x, y) = N(x-y) + K(x, y)$, *where* $N(x-y) = -\dfrac{1}{2\pi}\log|x-y|$ *and* $K \in C^2(B \times B)$;

(ii) $G(x, y) = G(y, x)$ *for* $x, y \in \bar{B}$;

(iii) $|\nabla_x G(x, y)| \le \dfrac{C}{|x-y|}$ *on* $B \times B$ *for some* $C > 0$.

Lemma 2.3 [7],[15],[16] *Let* $u \in C^2(\bar{\Omega})$ *satisfy*

$$\begin{cases} -\Delta u + \rho u = f, & x \in B; \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial B, \end{cases}$$

with $f \in C^0(\bar{\Omega})$, $\rho > 0$. *Then there exist positive constants C_m and C_n such that*

$$\|u\|_{L^m(\Omega)} \le C_m \|f\|_{L^1(\Omega)}, \quad 1 \le m < \infty,$$

$$\|\nabla u\|_{L^n(\Omega)} \le C_n \|f\|_{L^1(\Omega)}, \quad 1 \le n < 2.$$

We define the function $\phi \in C^1([0, \infty)) \cap W^{2,\infty}((0, \infty))$ *with* $r_2 > r_1 > 0$ *by*

$$\phi(r) := \begin{cases} r^2, & \text{if } 0 \le r \le r_1; \\ a_1 r^2 + a_2 r + a_3, & \text{if } r_1 \le r \le r_2; \\ r_1 r_2, & \text{if } r \ge r_2, \end{cases} \tag{2.1}$$

where $a_1 = -\dfrac{r_1}{r_2 - r_1}$, $a_2 = \dfrac{2r_1 r_2}{r_2 - r_1}$, *and* $a_3 = -\dfrac{r_1^2 r_2}{r_2 - r_1}$.

Lemma 2.4 [7],[15],[16] *We construct* $\Phi(x) := \phi(|x|) \in C^1(\mathbf{R}^2) \bigcap W^{2,\infty}(\mathbf{R}^2)$, *which satisfies the following*

$$\nabla \Phi(x) := \begin{cases} 2x, & \text{if } |x| \le r_1; \\ \dfrac{2r_1}{r_2 - r_1}(r_2 - |x|)\dfrac{x}{|x|}, & \text{if } r_1 \le |x| \le r_2; \\ 0, & \text{if } |x| \ge r_2, \end{cases}$$

and

$$|\nabla \Phi(x)| \le 2(\Phi(x))^{\frac{1}{2}}, \tag{2.2}$$

$$\Delta \Phi(x) = 4 \qquad \text{for } |x| \le r_1, \tag{2.3}$$

$$\Delta \Phi(x) \le 2 \qquad \text{for } |x| > r_1. \tag{2.4}$$

Moreover

$$\{\nabla \Phi(x) - \nabla \Phi(y)\} \cdot \nabla N(x-y) = -\frac{1}{\pi}, \quad (x,\, y) \in B_1 \times B_1, \tag{2.5}$$

$$\{\nabla \Phi(x) - \nabla \Phi(y)\} \cdot \nabla N(x-y) \leq \frac{2r_2 + r_1}{\pi(r_2 - r_1)}, \quad (x,\, y) \in \mathbf{R}^2 \times \mathbf{R}^2, \tag{2.6}$$

where $B_i := \{x \in \mathbf{R}^2 \mid |x| < r_i\}$ *with* $R,\, r_i > 0,\ i = 1, 2, 3, 4.$

3 Global Boundedness

Lemma 3.1 [22] *Let* $(u,\, v,\, w)$ *be a nonnegative local solution to* (1.4) *ensured by Lemma* 2.1. *Then for any* $\eta > 0,\ \theta > 1$, *there is* $c_1 = c_1(\eta,\, \theta) > 0$ *such that*

$$\int_\Omega v^\theta \leq \eta \int_\Omega u^{q\theta} + c_1 \|u_0\|^{q\theta}, \qquad t \in (0,\, T_{\max}) \tag{3.1}$$

and

$$\int_\Omega w^\theta \leq \eta \int_\Omega u^{r\theta} + c_1 \|u_0\|^{r\theta}, \qquad t \in (0,\, T_{\max}). \tag{3.2}$$

Proof. We integrate the first equation of (1.4) with respect to $x \in \Omega$ and obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u \mathrm{d}x &= \int_\Omega (\Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w)) \mathrm{d}x \\ &= \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu} - \chi u \frac{\partial v}{\partial \nu} + \xi u \frac{\partial w}{\partial \nu} \right) \mathrm{d}s \\ &= 0, \end{aligned}$$

so

$$\int_\Omega u \mathrm{d}x = \int_\Omega u_0 \mathrm{d}x. \tag{3.3}$$

The second equation of (1.4) implies

$$\|v\|_{L^1(\Omega)} = \frac{\alpha}{\beta} \|u\|^q_{L^q(\Omega)}. \tag{3.4}$$

Multiply the second equation of (1.4) by $v^{\theta-1}$, then integrate over Ω and by Young's inequality, we get

$$\begin{aligned} \frac{4(\theta-1)}{\theta^2} \int_\Omega |\nabla v^{\frac{\theta}{2}}|^2 + \beta \int_\Omega v^\theta &= \alpha \int_\Omega u^q v^{\theta-1} \\ &\leq \frac{\theta-1}{\theta} \beta \int_\Omega v^\theta + \frac{\alpha^\theta}{\theta \beta^{\theta-1}} \int_\Omega u^{q\theta}, \qquad t \in (0,\, T_{\max}). \end{aligned}$$

Then we can know that

$$\int_\Omega v^\theta \leq \frac{\alpha^\theta}{\beta^\theta} \int_\Omega u^{q\theta}, \quad \frac{4(\theta-1)}{\theta} \int_\Omega |\nabla v^{\frac{\theta}{2}}|^2 \leq \frac{\alpha^\theta}{\beta^{\theta-1}} \int_\Omega u^{q\theta}, \qquad t \in (0,\, T_{\max}). \tag{3.5}$$

In view of Ehrling's lemma, for any $\eta > 0,\ \theta > 1$, there exists a $c_2 = c_2(\eta,\, \theta) > 0$ such that

$$\|\Psi\|^2_{L^2(\Omega)} \leq \frac{\eta}{2} C_0 \|\Psi\|^2_{W^{1,2}(\Omega)} + c_2 \|\Psi\|^2_{L^{\frac{2}{\theta}}(\Omega)}, \qquad \Psi \in W^{1,2}(\Omega) \tag{3.6}$$

with $C_0 = \dfrac{\beta^\theta}{\alpha^\theta} \dfrac{4(\theta-1)}{4(\theta-1) + \beta^\theta}$. Let $\Psi = v^{\frac{\theta}{2}}$. We know from (3.4)–(3.6) that

$$\int_\Omega v^\theta \leq \frac{\eta}{2} \int_\Omega u^{q\theta} + c_3 \|u\|^{q\theta}_{L^q(\Omega)} \tag{3.7}$$

with $c_3 = c_3(\eta, \theta) > 0$. Since $1 \le q \le q\theta$, by the interpolation inequality, the Young's inequality and (3.3), we have

$$\|u\|^{q\theta}_{L^q(\Omega)} \le \|u\|^{q\theta\tau}_{L^q(\Omega)} \|u\|^{q\theta(1-\tau)}_{L^1(\Omega)} \le \frac{\eta}{2c_3} \int_\Omega u^{q\theta} + c_4 \left(\int_\Omega u_0 \right)^{q\theta}, \tag{3.8}$$

where $\tau = \dfrac{q-1}{q-\frac{1}{\theta}} \in (0, 1)$, and $c_4 = c_4(\eta, \theta) > 0$. Combining (3.7) and (3.8), we conclude (3.1).

Similarly, (3.2) can be obtained by the same procedure as above. This completes the proof.

Proof of Theorem 1.1 We begin with showing that for any $p > 1$, there exists a $c = c(p) > 0$ such that

$$\int_\Omega u^p \le c, \quad t \in (0, T_{\max}). \tag{3.9}$$

Multiply the first equation of (1.4) by u^{p-1} and then integrate over Ω, we obtain that

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p =& -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - \frac{\chi(p-1)}{p}\int_\Omega u^p \Delta v + \frac{\xi(p-1)}{p}\int_\Omega u^p \Delta w \\ \le& -\frac{\chi(p-1)}{p}\int_\Omega u^p \Delta v + \frac{\xi(p-1)}{p}\int_\Omega u^p \Delta w \\ =& \frac{\alpha\chi(p-1)}{p}\int_\Omega u^{p+q} - \frac{\beta\chi(p-1)}{p}\int_\Omega u^p v - \\ & \frac{\xi\gamma(p-1)}{p}\int_\Omega u^{p+r} + \frac{\xi\delta(p-1)}{p}\int_\Omega u^p w, \quad t \in (0, T_{\max}). \end{aligned} \tag{3.10}$$

By Young's inequality with (3.2) and $\eta > 0$, we have

$$\begin{aligned} \frac{\xi\delta(p-1)}{p}\int_\Omega u^p w \le& \frac{\eta}{2}\int_\Omega u^{p+r} + c_5 \int_\Omega w^{\frac{p+r}{r}} \\ \le& \eta \int_\Omega u^{p+r} + c_6, \quad t \in (0, T_{\max}), \end{aligned}$$

where $c_5 = c_5(p, \eta) > 0$, $c_6 = c_6(p, \eta) > 0$. Hence

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le \frac{\chi\alpha(p-1)}{p}\int_\Omega u^{p+q} - \left(\frac{\xi\gamma(p-1)}{p} - \eta\right)\int_\Omega u^{p+r} + c_6, \quad t \in (0, T_{\max}). \tag{3.11}$$

Case 1. Let $q < r$, by Young's inequality, we have

$$\frac{\chi\alpha(p-1)}{p}\int_\Omega u^{p+q} \le \frac{\xi\gamma(p-1)}{2p}\int_\Omega u^{p+r} + c_7, \quad t \in (0, T_{\max}) \tag{3.12}$$

with $c_7 = c_7(p) > 0$. We know from (3.11) and (3.12) that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le -\left(\frac{\xi\gamma(p-1)}{2p} - \eta\right)\int_\Omega u^{p+r} + c_8, \quad t \in (0, T_{\max}),$$

where $c_8 = c_6 + c_7$. We let $\eta = \dfrac{\xi\gamma(p-1)}{4p}$, hence

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le -\frac{\xi\gamma(p-1)}{4p}\int_\Omega u^{p+r} + c_8, \quad t \in (0, T_{\max}). \tag{3.13}$$

Due to Young's inequality, there is a $c_9 = c_9(p) > 0$ such that

$$\int_\Omega u^p \le \frac{\xi\gamma(p-1)}{4p}\int_\Omega u^{p+r} + c_9, \quad t \in (0, T_{\max}). \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le -\int_\Omega u^p + c_{10}, \quad t \in (0, T_{\max})$$

with $c_{10} = c_8 + c_9$. This implies (3.9).

Case 2. If $q = r$, we know from (3.11) that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le \left(\frac{(\chi\alpha - \xi\gamma)(p-1)}{p} + \eta\right)\int_\Omega u^{p+q} + c_6, \quad t \in (0, T_{\max}).$$

If $\chi\alpha - \xi\gamma < 0$, taking $\eta = -\dfrac{(\chi\alpha - \xi\gamma)(p-1)}{2p} > 0$, we have that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le \frac{(\chi\alpha - \xi\gamma)(p-1)}{2p}\int_\Omega u^{p+q} + c_6, \quad t \in (0, T_{\max}). \tag{3.15}$$

By Young's inequality, there exists $c_{11} = c_{11}(p) > 0$,

$$\int_\Omega u^p \le -\frac{(\chi\alpha - \xi\gamma)(p-1)}{2p}\int_\Omega u^{p+r} + c_{11}, \quad t \in (0, T_{\max}). \tag{3.16}$$

We get from (3.15) and (3.16) that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le -\int_\Omega u^p + c_{12}, \quad t \in (0, T_{\max})$$

with $c_{12} = c_6 + c_{11}$. This implies (3.9).

Now, we show that

$$\|u\|_{L^\infty(\Omega)} \le C, \quad t \in (0, T_{\max}) \tag{3.17}$$

with some $C > 0$, which concludes $T_{\max} = \infty$ by Lemma 2.1.

Case 1. Let $p_0 > \max\{rn, 1\}$. Applying an elliptic L^p estimate to the third equation in (1.4) we get from (3.9) that

$$\|w(\cdot, t)\|_{w^{2,\frac{p_0}{r}}} \le C_0, \quad t \in (0, T_{\max})$$

with $C_0 > 0$. Then by the Sobolev imbedding theorem with $c_0 > 0$, we have that

$$\|w(\,\cdot\,, t)\|_{L^\infty(\Omega)} \le c_0, \quad t \in (0, T_{\max}). \tag{3.18}$$

We get from (3.10) and (3.18) that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le \frac{\chi\alpha(p-1)}{p}\int_\Omega u^{p+q} - \frac{\xi\gamma(p-1)}{p}\int_\Omega u^{p+r} + \frac{\xi\delta c_0(p-1)}{p}\int_\Omega u^p, \quad t \in (0, T_{\max}). \tag{3.19}$$

Case 2. Let $q < r$. By Young's inequality, we have that

$$\begin{aligned} &\frac{\chi\alpha(p-1)}{p}\int_\Omega u^{p+q} \\ \le\ &\frac{\xi\gamma(p-1)}{2p}\int_\Omega u^{p+r} + \frac{r-q}{p+r}\left(\frac{\chi\alpha(p-1)}{p}\right)^{\frac{p+r}{r-q}}\left(\frac{p+r}{p+q}\frac{\xi\gamma(p-1)}{2p}\right)^{-\frac{p+q}{r-q}}|\Omega|, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\frac{\xi\delta c_0(p-1)}{p}\int_\Omega u^p \\ \le\ &\frac{\xi\gamma(p-1)}{2p}\int_\Omega u^{p+r} + \frac{r}{p+r}\left(\frac{p+r}{p}\frac{\xi\gamma(p-1)}{2p}\right)^{-\frac{p}{r+p}}\left(\frac{\xi\delta c_0(p-1)}{p}\right)^{\frac{p+r}{r}}|\Omega|. \end{aligned} \tag{3.21}$$

Combining (3.19)–(3.21), we have that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le \frac{r-q}{p+r}\left(\frac{\chi\alpha(p-1)}{p}\right)^{\frac{p+r}{r-q}}\left(\frac{p+r}{p+q}\frac{\xi\gamma(p-1)}{2p}\right)^{-\frac{p+q}{r-q}}|\Omega|+ \frac{r}{p+r}\left(\frac{p+r}{p}\frac{\xi\gamma(p-1)}{2p}\right)^{-\frac{p}{r+p}}\left(\frac{\xi\delta c_0(p-1)}{p}\right)^{\frac{p+r}{r}}|\Omega|,\quad t\in(0,\,T_{\max}),$$

and then

$$\begin{aligned}\|u\|_{L^P(\Omega)} \le{}& \|u_0\|_{L^P(\Omega)} + \left[\frac{r-q}{p+r}\left(\frac{\chi\alpha(p-1)}{p}\right)^{\frac{p+r}{r-q}}\left(\frac{p+r}{p+q}\frac{\xi\gamma(p-1)}{2p}\right)^{-\frac{p+q}{r-q}}|\Omega|t\right]^{\frac{1}{p}}+\\ &\left[\frac{r}{p+r}\left(\frac{p+r}{p}\frac{\xi\gamma(p-1)}{2p}\right)^{-\frac{p}{r+p}}\left(\frac{\xi\delta c_0(p-1)}{p}\right)^{\frac{p+r}{r}}|\Omega|t\right]^{\frac{1}{p}}\\ =:{}& \|u_0\|_{L^p(\Omega)} + F(p,\,t),\quad t\in(0,\,T_{\max}),\ p>1.\end{aligned}$$

For any $t\in(0,\,T_{\max})$, we have that

$$\lim_{p\to\infty} F(p,\,t) = (\chi\alpha)^{\frac{1}{r-q}} + (\delta\xi c_0)^{\frac{1}{r}},$$

and thus (3.17) is obtained.

Let $q=r$, (3.19) becomes that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le \frac{(\chi\alpha-\xi\gamma)(p-1)}{p}\int_\Omega u^{p+r} + \frac{\xi\delta c_0(p-1)}{p}\int_\Omega u^p,\quad t\in(0,\,T_{\max}). \tag{3.22}$$

By Young's inequality and $\chi\alpha-\xi\gamma<0$, we know that

$$\frac{\xi\delta c_0(p-1)}{p}\int_\Omega u^p \le \frac{(\xi\gamma-\chi\alpha)(p-1)}{p}\int_\Omega u^{p+r}+ \frac{r}{p+r}\left[\left(\frac{p+r}{p}\frac{(\xi\gamma-\chi\alpha)(p-1)}{p}\right)^{-\frac{p}{r+p}}\frac{\xi\delta c_0(p-1)}{p}\right]^{\frac{p+r}{r}}|\Omega|. \tag{3.23}$$

From (3.22) and (3.33), we can obtain that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p \le \frac{r}{p+r}\left[\left(\frac{p+r}{p}\frac{(\xi\gamma-\chi\alpha)(p-1)}{p}\right)^{-\frac{p}{r+p}}\frac{\xi\delta c_0(p-1)}{p}\right]^{\frac{p+r}{r}}|\Omega|,$$

and hence

$$\begin{aligned}\|u\|_{L^P(\Omega)} \le{}& \|u_0\|_{L^P(\Omega)} + \left\{\frac{r}{p+r}\left[\left(\frac{p+r}{p}\frac{(\xi\gamma-\chi\alpha)(p-1)}{p}\right)^{-\frac{p}{r+p}}\frac{\xi\delta c_0(p-1)}{p}\right]^{\frac{p+r}{r}}|\Omega|t\right\}^{\frac{1}{p}}\\ =:{}& \|u_0\|_{L^P(\Omega)} + H(p,t).\end{aligned}$$

Obviously,

$$\lim_{p\to\infty} H(p,\,t) = (\delta\xi C_0)^{\frac{1}{r}},\quad t\in(0,\,T_{\max}).$$

Thus (3.17) is obtained.

The proof of Theorem 1.1 is completed.

4 Blow-up of Nonradial Solutions

Let $(u,\,v,\,w)$ be the local solution of (1.4) ensured by Lemma 2.1. We should show $T_{\max}<\infty$. It suffices to find a $T>0$ such that the Φ-weighted integral of $u(x,\,t)$ tends to zeros as $t\to T$. Inspired by [7], [15], [16], this is realized by the following propositions.

Proposition 4.1 *Let $q = r$, $x_0 \in \Omega$ and $0 < r_1 < r_2 < \mathrm{dist}(x_0, \partial\Omega)$, where $\mathrm{dist}(x_0, \partial\Omega)$ denotes the distance between x_0 and $\partial\Omega$. Then there exist $C_1, C_2 > 0$ relying on r_1, r_2, $\mathrm{dist}(x_0, \partial\Omega)$ such that for $t \in (0, T_{\max})$,*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x, t)\Phi(x-x_0)\mathrm{d}x\\
\le\ & 4\int_\Omega u_0(x)\mathrm{d}x - \frac{(\chi\alpha-\xi\gamma)}{2\pi|\Omega|^{r-1}}\left(\int_\Omega u_0(x)\mathrm{d}x\right)^{r+1}+\\
& C_1\int_\Omega u^r(x, t)\mathrm{d}x\left(\int_\Omega u(x, t)\Phi(x-x_0)\mathrm{d}x\right)+\\
& C_2\int_\Omega u^r(x, t)\mathrm{d}x\left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac12}\left(\int_\Omega u(x, t)\Phi(x-x_0)\mathrm{d}x\right)^{\frac12}+\\
& C_3\int_\Omega u_0(x)\mathrm{d}x\left(\int_\Omega u^{2r-1}(x, t)\mathrm{d}x\right)^{\frac12}\left(\int_\Omega u(x, t)\Phi(x-x_0)\mathrm{d}x\right)^{\frac12},
\end{aligned}\tag{4.1}$$

where $\Phi(x) = \phi(|x|)$ defined by Lemma 2.1.

Proof. We proceed somewhat similarly to [7] and [16]. Without loss of generality, we suppose that x_0 is the origin. Multiply the first equation of (1.4) by $\Phi(x)$ and integrate over Ω. Since the Neumann boundary condition of (1.4), $\left.\frac{\partial\Phi}{\partial\nu}\right|_{\partial\Omega} = 0$ with $r_2 < \mathrm{dist}(x_0, \partial\Omega)$ by Lemma 2.4, we know that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x, t)\Phi(x)\mathrm{d}x =\ & \int_\Omega \Phi(x)(\Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w))\mathrm{d}x\\
=\ & \int_\Omega u(x, t)\Delta\Phi(x) + \chi\int_\Omega u(x, t)\nabla\Phi(x)\cdot\nabla v(x, t)\mathrm{d}x-\\
& \xi\int_\Omega u(x, t)\nabla\Phi(x)\cdot\nabla w(x, t)\mathrm{d}x.
\end{aligned}$$

By $\Delta\Phi \le 4$ on Ω (see (2.3) and (2.4)) and $\int_\Omega u(x)\mathrm{d}x = \int_\Omega u_0(x)\mathrm{d}x$, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x, t)\Phi(x)\mathrm{d}x \le\ & 4\int_\Omega u_0(x)\mathrm{d}x + \chi\int_\Omega u(x, t)\nabla\Phi(x)\cdot\nabla v(x, t)\mathrm{d}x-\\
& \xi\int_\Omega u(x, t)\nabla\Phi(x)\cdot\nabla w(x, t)\mathrm{d}x\\
=:\ & 4\int_\Omega u_0(x)\mathrm{d}x + Z_1 + Z_2.
\end{aligned}\tag{4.2}$$

Take $r_3, r_4 > 0$ such that $0 < r_2 < r_3 < r_4 < \mathrm{dist}(x_0, \partial\Omega)$ and define $\zeta \in C_0^\infty(\mathbf{R}^2)$ with $0 \le \zeta \le 1$ satisfying

$$\zeta(x) = \begin{cases} 1, & \text{if } |x| \le r_3,\\ 0, & \text{if } |x| \ge r_4,\end{cases}$$

and $h(x, t) := \zeta(x)v(x, t)$. By the second equation of (1.4), it is easy that we can verify that $h(x, t)$ satisfies

$$\begin{cases} -\Delta h = \alpha\zeta u^q + g - \beta\zeta v, & x \in B_4,\\ h = 0, & x \in \partial B_4\end{cases}$$

for $t \in (0, T_{\max})$, where $g := -2\nabla\zeta\cdot\nabla v - \Delta\zeta v$. By Lemma 2.2, $h(x, t)$ can be expressed as follows

$$h(x, t) = \int_{B_4} G(x, y)\{\alpha\zeta(y)u^q(y, t) + g(y, t) - \beta\zeta v(y, t)\}\mathrm{d}y.$$

Since $\nabla\Phi \equiv 0$ outside of B_2 by Lemma 2.4 and $h \equiv v$ in B_3, we can obtain that

$$\begin{aligned} Z_1 &= \chi\int_\Omega u(x, t)\nabla\Phi(x)\cdot\nabla v(x, t)\mathrm{d}x \\ &= \chi\alpha\int_{B_2}\int_{B_4} u(x, t)u^q(y, t)\zeta(y)\nabla\Phi(x)\cdot\nabla_x G(x, y)\mathrm{d}y\mathrm{d}x + \\ &\quad \chi\int_{B_2}\int_{B_4} u(x, t)g(y)\nabla\Phi(x)\cdot\nabla_x G(x, y)\mathrm{d}y\mathrm{d}x - \\ &\quad \chi\beta\int_{B_2}\int_{B_4} u(x, t)\zeta(y)v(y, t)\nabla\Phi(x)\cdot\nabla_x G(x, y)\mathrm{d}y\mathrm{d}x \\ &=: I + II + III. \end{aligned}$$

In the following we firstly estimate on I. Since Lemma 2.2(i) with $\zeta = 1$ in B_3,

$$\begin{aligned} I &= \chi\alpha\int_{B_2}\int_{B_4} u(x, t)u^q(y, t)\zeta(y)\nabla\Phi(x)\cdot\nabla_x G(x, y)\mathrm{d}y\mathrm{d}x \\ &= \chi\alpha\int_{B_2}\int_{B_3} u(x, t)u^q(y, t)\nabla\Phi(x)\cdot\nabla_x N(x-y)\mathrm{d}y\mathrm{d}x + \\ &\quad \chi\alpha\int_{B_2}\int_{B_4\setminus B_3} u(x, t)u^q(y, t)\zeta(y, t)\nabla\Phi(x)\cdot\nabla_x N(x-y)\mathrm{d}y\mathrm{d}x + \\ &\quad \chi\alpha\int_{B_2}\int_{B_4} u(x, t)u^q(y, t)\zeta(y)\nabla\Phi(x)\cdot\nabla_x K(x-y)\mathrm{d}y\mathrm{d}x \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Because of the symmetry property of integrals and $\nabla\Phi \equiv 0$ outside B_2, we get from the estimate (2.5), (2.6) and $\Phi \ge r_1^2$ outside B_1 that

$$\begin{aligned} I_1 &= \chi\alpha\int_{B_2}\int_{B_3} u(x, t)u^q(y, t)\nabla\Phi(x)\cdot\nabla_x N(x-y)\mathrm{d}y\mathrm{d}x \\ &= \frac{\chi\alpha}{2}\int_{B_3}\int_{B_3} u(x, t)u^q(y, t)\{\nabla\Phi(x) - \nabla\Phi(y)\}\cdot\nabla_x N(x-y)\mathrm{d}y\mathrm{d}x \\ &\le \frac{\chi\alpha}{2}\int_{B_1}\int_{B_1} u(x, t)u^q(y, t)\{\nabla\Phi(x) - \nabla\Phi(y)\}\cdot\nabla_x N(x-y)\mathrm{d}y\mathrm{d}x + \\ &\quad \frac{\chi\alpha}{2}\iint_{(B_3\times B_3)\setminus(B_1\times B_1)} u(x, t)u^q(y, t)|\{\nabla\Phi(x) - \nabla\Phi(y)\}\cdot\nabla_x N(x-y)|\mathrm{d}y\mathrm{d}x \\ &\le -\frac{\chi\alpha}{2\pi}\int_{B_1}\int_{B_1} u(x, t)u^q(y, t)\mathrm{d}y\mathrm{d}x + \\ &\quad \frac{\chi\alpha(2r_2 + r_1)}{2\pi(r_2 - r_1)}\iint_{(B_3\times B_3)\setminus(B_1\times B_1)} u(x, t)u^q(y, t)\mathrm{d}y\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} &-\frac{\chi\alpha}{2\pi}\int_{B_1}\int_{B_1} u(x, t)u^q(y, t)\mathrm{d}y\mathrm{d}x \\ = &-\frac{\chi\alpha}{2\pi}\int_{B_1} u(x, t)\mathrm{d}x\int_{B_1} u^q(x, t)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\chi\alpha}{2\pi}\left(\int_\Omega u(x,t)\mathrm{d}x-\int_{\Omega\setminus B_1}u(x,t)\mathrm{d}x\right)\left(\int_\Omega u^q(x,t)\mathrm{d}x-\int_{\Omega\setminus B_1}u^q(x,t)\mathrm{d}x\right)\\
&\le -\frac{\chi\alpha}{2\pi}\int_\Omega u(x,t)\mathrm{d}x\int_\Omega u^q(x,t)\mathrm{d}x+\frac{\chi\alpha}{2\pi}\int_\Omega u(x,t)\mathrm{d}x\int_{\Omega\setminus B_1}u^q(x,t)\mathrm{d}x+\\
&\quad\frac{\chi\alpha}{2\pi}\int_{\Omega\setminus B_1}u(x,t)\mathrm{d}x\int_\Omega u^q(x,t)\mathrm{d}x\\
&\le -\frac{\chi\alpha}{2\pi}\int_\Omega u_0(x)\mathrm{d}x\int_\Omega u^q(x,t)\mathrm{d}x+\frac{\chi\alpha}{2\pi r_1}\int_\Omega u_0(x)\mathrm{d}x\int_\Omega u^q(x,t)\Phi^{\frac{1}{2}}(x)\mathrm{d}x+\\
&\quad\frac{\chi\alpha}{2\pi r_1^2}\int_\Omega u^q(x,t)\mathrm{d}x\int_\Omega u(x,t)\Phi(x)\mathrm{d}x\\
&\le -\frac{\chi\alpha}{2\pi}\int_\Omega u_0(x)\mathrm{d}x\int_\Omega u^q(x,t)\mathrm{d}x+\\
&\quad\frac{\chi\alpha}{2\pi r_1}\int_\Omega u_0(x)\mathrm{d}x\left(\int_\Omega u^{2q-1}(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}+\\
&\quad\frac{\chi\alpha}{2\pi r_1^2}\int_\Omega u^q(x)\mathrm{d}x\int_\Omega u(x,t)\Phi(x)\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\chi\alpha(2r_2+r_1)}{2\pi(r_2-r_1)}\iint_{(B_3\times B_3)\setminus(B_1\times B_1)}u(x,t)u^q(y,t)\mathrm{d}y\mathrm{d}x\\
&\quad\le\frac{\chi\alpha(2r_2+r_1)}{2\pi r_1^2(r_2-r_1)}\int_\Omega u^q(x,t)\mathrm{d}x\int_\Omega u(x,t)\Phi(x)\mathrm{d}x.
\end{aligned}
$$

So

$$
\begin{aligned}
I_1\le&-\frac{\chi\alpha}{2\pi}\int_\Omega u_0(x)\mathrm{d}x\int_\Omega u^q(x)\mathrm{d}x+\\
&\frac{\chi\alpha}{2\pi r_1}\int_\Omega u(x,t)\mathrm{d}x\left(\int_\Omega u^{2q-1}(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}+\\
&\frac{\chi\alpha 3r_2}{2\pi r_1^2(r_2-r_1)}\int_\Omega u^q(x,t)\mathrm{d}x\int_\Omega u(x,t)\Phi(x)\mathrm{d}x.
\end{aligned}
$$

Using $|\nabla_x N(x-y)|=\dfrac{1}{2\pi}\cdot\dfrac{1}{|x-y|}$ from Lemma 2.2(iii), $|x-y|\ge r_3-r_2$ for $(x,y)\in B_2\times(B_4\setminus B_3)$, by the estimate (2.2) with the Hölder's inequality, we have that

$$
\begin{aligned}
I_2&=\chi\alpha\int_{B_2}\int_{B_4\setminus B_3}u(x,t)u^q(y,t)\zeta(y)\nabla\Phi(x)\cdot\nabla_x N(x-y)\mathrm{d}y\mathrm{d}x\\
&\le\frac{\chi\alpha}{2\pi}\int_{B_2}\int_{B_4\setminus B_3}u(x,t)u^q(y,t)|\nabla\Phi(x)|\cdot\frac{1}{|x-y|}\mathrm{d}y\mathrm{d}x\\
&\le\frac{\chi\alpha}{\pi(r_3-r_2)}\int_{B_2}\int_{B_4\setminus B_3}u(x,t)u^q(y,t)(\Phi(x))^{\frac{1}{2}}\mathrm{d}y\mathrm{d}x\\
&\le\frac{\chi\alpha}{\pi(r_3-r_2)}\int_\Omega u^q(x,t)\mathrm{d}x\int_\Omega u(x,t)^{\frac{1}{2}}(u(x,t)\Phi(x))^{\frac{1}{2}}\mathrm{d}x\\
&\le\frac{\chi\alpha}{\pi(r_3-r_2)}\int_\Omega u^q(x,t)\mathrm{d}x\left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned}
$$

According to Lemma 2.2(ii), I_3 can be expressed as follows:

$$\begin{aligned} I_3 &= \chi\alpha \int_{B_2}\int_{B_4} u(x,\,t)u^q(y,\,t)\zeta(y)\nabla\Phi(x)\cdot\nabla_x K(x-y)\mathrm{d}y\mathrm{d}x \\ &\le 2\chi\alpha\|\nabla_x k\|_{L^\infty(B_2\times B_4)} \int_{B_2}\int_{B_4} u(x,\,t)u^q(y,\,t)(\Phi(x))^{\frac12}\mathrm{d}y\mathrm{d}x \\ &\le 2\chi\alpha\|\nabla_x k\|_{L^\infty(B_2\times B_4)} \int_{\Omega} u^q(x,\,t)\mathrm{d}x \int_{\Omega} u^{\frac12}(x,\,t)(u(x,\,t)\Phi(x))^{\frac12}\mathrm{d}x \\ &\le 2\chi\alpha\|\nabla_x k\|_{L^\infty(B_2\times B_4)} \int_{\Omega} u^q(x,\,t)\mathrm{d}x \left(\int_{\Omega} u_0(x)\mathrm{d}x\right)^{\frac12}\left(\int_{\Omega}(u(x,\,t)\Phi(x))\mathrm{d}x\right)^{\frac12}. \end{aligned}$$

Thus we obtain the estimate

$$\begin{aligned} I \le\ & -\frac{\chi\alpha}{2\pi}\int_{\Omega} u_0(x)\mathrm{d}x \int_{\Omega} u^q(x,\,t)\mathrm{d}x + \\ & \frac{\chi\alpha}{2\pi r_1}\int_{\Omega} u_0(x)\mathrm{d}x \left(\int_{\Omega} u^{2q-1}(x,\,t)\mathrm{d}x\right)^{\frac12}\left(\int_{\Omega} u(x,\,t)\Phi(x)\mathrm{d}x\right)^{\frac12} + \\ & \frac{\chi\alpha 3r_2}{2\pi r_1(r_2-r_1)}\int_{\Omega} u^q(x,\,t)\mathrm{d}x \int_{\Omega} u(x,\,t)\Phi(x)\mathrm{d}x + \\ & \left(\frac{\chi\alpha}{\pi(r_3-r_2)} + 2\chi\alpha\|\nabla_x k\|_{L^\infty(B_2\times B_4)}\right)\int_{\Omega} u^q(x,\,t)\mathrm{d}x\cdot \\ & \left(\int_{\Omega} u_0(x)\mathrm{d}x\right)^{\frac12}\left(\int_{\Omega}(u(x,\,t)\Phi(x))\mathrm{d}x\right)^{\frac12}. \end{aligned}$$

Now, we estimate II. Since $g := -2\nabla\zeta\cdot\nabla v - \triangle\zeta v$ in B_3, we obtain by Lemma 2.2(iii) and the Hölder's inequality that

$$\begin{aligned} II &= \chi \int_{B_2}\int_{B_4\setminus B_3} u(x,\,t)g(y,\,t)\nabla\Phi(x)\cdot\nabla_x G(x,\,y)\mathrm{d}y\mathrm{d}x \\ &\le \frac{C_3\chi}{r_3-r_2}\int_{B_2}\int_{B_4\setminus B_3} u(x,\,t)|g(y,\,t)||\nabla\Phi(x)|\mathrm{d}y\mathrm{d}x \\ &\le \frac{C_4}{r_3-r_2}\left(\int_{\Omega} u(x,\,t)\Phi^{\frac12}(x)\mathrm{d}x\right)\left(\int_{\Omega}|g(y,\,t)|\mathrm{d}y\right) \end{aligned}$$

with $C_3,\, C_4 > 0$.

To calculate further, we apply Lemma 2.3 to the third equation of (1.4) to get

$$\|v\|_{w^{1,1}(\Omega)} \le C_5\|u^q\|_{L^1(\Omega)}$$

with C_5 is a positive constant. Then

$$\int_{\Omega}|g(y,\,t)|\mathrm{d}y \le 2(\|\nabla\zeta\|_{L^\infty} + \|\triangle\zeta\|_{L^\infty})\|v\|_{w^{1,1}(\Omega)} \le C_6\int_{\Omega} u^q(x,\,t)\mathrm{d}x,$$

where C_6 is a constant depending on $\|\nabla\zeta\|_{L^\infty}$, $\|\triangle\zeta\|_{L^\infty}$ and C_5. Hence

$$II \le \frac{C_4C_6}{r_3-r_2}\int_{\Omega} u^q(x,\,t)\mathrm{d}x\left(\int_{\Omega} u_0(x)\mathrm{d}x\right)^{\frac12}\left(\int_{\Omega} u(x,t)\Phi(x)\mathrm{d}x\right)^{\frac12}.$$

Estimating III, letting $\psi(x,\,t) := -\int_{B_4}\zeta(y)v(y,\,t)G(x,\,t)\mathrm{d}y$, we have that

$$\begin{aligned} III &= \chi\beta\int_{B_2}\int_{B_4} u(x,\,t)\zeta(y,\,t)v(y,\,t)\nabla\Phi(x)\cdot\nabla_x G(x,\,y)\mathrm{d}y\mathrm{d}x \\ &= \chi\beta\int_{B_2} u(x,\,t)\nabla\Phi(x)\cdot\nabla\psi(x,\,t)\mathrm{d}y\mathrm{d}x. \end{aligned}$$

Noticing $|\nabla_x G(x-y)| \le C_7|x-y|$ and using the Hölder's inequality, we obtain that

$$\begin{aligned}|\nabla\psi(x,\,t)| &\le C_7\int_{B_4}\frac{v(y,\,t)}{|x-y|}\mathrm{d}y\\ &\le C_7\left(\int_{B_4}v^2(y,\,t)\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{B_4}|x-y|^{-2}\mathrm{d}y\right)^{\frac{1}{2}}\\ &\le C_7\|v\|_{L^2(\Omega)}\end{aligned}$$

with $C_7 > 0$. Here, we use

$$\sup_{x\in B_4}\left(\int_{B_4}|x-y|^{-2}\mathrm{d}y\right)^{\frac{1}{2}} \le \infty.$$

Applying Lemma 2.3 to the third equation of (1.4), we have that

$$\|v(t)\|_{L^2(\Omega)} \le C_8\|u^q\|_{L^1(\Omega)}$$

with $C_8 > 0$. Hence

$$|\nabla\psi(x,\,t)| \le C_9\int_\Omega u^q(x,\,t)\mathrm{d}x, \quad x\in B_4,$$

where $C_9 = C_7C_8$. Thus

$$III \le C_{10}\int_\Omega u^q(x,\,t)\mathrm{d}x\left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}$$

with $C_{10} > 0$.

Combining the estimating I, II, III, we can obtain

$$\begin{aligned}Z_1 :=&\ \chi\int_\Omega u(x,\,t)\nabla\Phi(x)\cdot\nabla v(x,\,t)\mathrm{d}x\\ \le&\ -\frac{\chi\alpha}{2\pi}\int_\Omega u^q(x,\,t)\int_\Omega u_0(x)\mathrm{d}x + \overline{C}_1\int_\Omega u^q(x,\,t)\mathrm{d}x\left(\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\right) +\\ &\ \overline{C}_2\int_\Omega u^q(x,\,t)\mathrm{d}x\left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}} +\\ &\ \overline{C}_3\int_\Omega u_0(x)\mathrm{d}x\left(\int_\Omega u^{2q-1}(x,\,t)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}\end{aligned} \tag{4.3}$$

with $\overline{C}_1$, $\overline{C}_2 > 0$, $\overline{C}_3 > 0$. By the procedure in the proof of [15], we have that

$$\begin{aligned}Z_2 :=&\ -\xi\int_\Omega u(x,\,t)\nabla\Phi(x)\cdot\nabla w(x,\,t)\mathrm{d}x\\ \le&\ \frac{\xi\gamma}{2\pi}\int_\Omega u^r(x)\mathrm{d}x\int_\Omega u_0(x)\mathrm{d}x + \widetilde{C}_1\int_\Omega u^r(x,\,t)\mathrm{d}x\left(\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\right) +\\ &\ \widetilde{C}_2\int_\Omega u^r(x,\,t)\mathrm{d}x\left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}\end{aligned} \tag{4.4}$$

with $\widetilde{C}_1$, $\widetilde{C}_2 > 0$.

Combining (4.2) with (4.3) and (4.4), we obtain that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\\ &\le 4\int_\Omega u_0(x)\mathrm{d}x - \frac{(\chi\alpha-\xi\gamma)}{2\pi}\int_\Omega u_0(x)\mathrm{d}x\int_\Omega u^r(x,\,t)\mathrm{d}x\\ &\quad + C_1\int_\Omega u^r(x,\,t)\mathrm{d}x\left(\int_\Omega u(x,\,t)\Phi(x)\mathrm{d}x\right)\end{aligned}$$

$$+ C_2 \int_\Omega u^r(x,t)\mathrm{d}x \left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u(x,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}$$
$$+ C_3 \int_\Omega u_0(x)\mathrm{d}x \left(\int_\Omega u^{2r-1}(x,t)\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u(x,t)\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}. \quad (4.5)$$

By the Hölder's inequality, one has

$$\int_\Omega u(x,t)\mathrm{d}x \le |\Omega|^{\frac{r-1}{r}} \left(\int_\Omega u^r(x,t)\mathrm{d}x\right)^{\frac{1}{r}},$$
$$|\Omega|^{1-r} \left(\int_\Omega u(x,t)\mathrm{d}x\right)^r \le \int_\Omega u^r(x,t)\mathrm{d}x.$$

If $\chi\alpha - \xi\gamma > 0$, we have that

$$-\frac{\chi\alpha - \xi\gamma}{2\pi|\Omega|^{r-1}} \left(\int_\Omega u(x,t)\mathrm{d}x\right)^r \ge -\frac{\chi\alpha - \xi\gamma}{2\pi} \int_\Omega u^r(x,t)\mathrm{d}x. \quad (4.6)$$

Combining (4.5) and (4.6), this completes the proof.

Lemma 4.1 *Let $\Omega \in \mathbf{R}^2$ be a smooth bounded domain and $x_0 \in \Omega$. Assume that $\int_\Omega u_0(x)|x-x_0|^2\mathrm{d}x$ is sufficiently small, $q = r$, $\chi\alpha - \xi\gamma > 0$, and*

$$\left(\int_\Omega u_0(x)\mathrm{d}x\right)^r \ge \frac{8\pi|\Omega|^{r-1}}{\chi\alpha - \xi\gamma}.$$

Then the solution of (1.4) *blows up in finite time.*

Proof. Denote

$$M_{\Phi(t)} := \int_\Omega u(x,t)\Phi(x-x_0)\mathrm{d}x.$$

We define

$$E(s) := 4\int_\Omega u_0(x)\mathrm{d}x - \frac{(\chi\alpha - \xi\gamma)}{2\pi|\Omega|^{r-1}} \left(\int_\Omega u_0(x)\mathrm{d}x\right)^{r+1} + C_1 \int_\Omega u^r(x,t)\mathrm{d}x s +$$
$$C_2 \int_\Omega u^r(x,t)\mathrm{d}x \left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}} s^{\frac{1}{2}} + C_3 \int_\Omega u_0(x)\mathrm{d}x \left(\int_\Omega u^{2r-1}(x,t)\mathrm{d}x\right)^{\frac{1}{2}} s^{\frac{1}{2}}.$$

For any $T < \infty$, if $\int_\Omega u^{2r-1}(x,t)\mathrm{d}x \to \infty$ as $t \to T$, then $\|u\|_{L^\infty} \to \infty$ as $t \to T$, our aim is obtained.

Otherwise, $\int_\Omega u^{2r-1}(x,t)\mathrm{d}x \le C(T)$, then

$$E(s) := 4\int_\Omega u_0(x)\mathrm{d}x - \frac{(\chi\alpha - \xi\gamma)}{2\pi|\Omega|^{r-1}} \left(\int_\Omega u_0(x)\mathrm{d}x\right)^{r+1} + \widetilde{c}_1 s +$$
$$\widetilde{c}_2 \left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}} s^{\frac{1}{2}} + \widetilde{c}_3 \int_\Omega u_0(x)\mathrm{d}x s^{\frac{1}{2}}$$

with $\widetilde{c}_1, \widetilde{c}_2, \widetilde{c}_3 > 0$, which are depended on T. Proposition 4.1 says

$$\frac{\mathrm{d}}{\mathrm{d}t} M_{\Phi(t)} \le E(M_{\Phi(t)}), \quad t \in (0, T_{\max}). \quad (4.7)$$

By the definition of $\Phi(x)$ with Lemma 2.3 associated, we know

$$M_{\Phi(0)} := \int_\Omega u_0(x,t)\Phi(x-x_0)\mathrm{d}x \le \int_\Omega u_0(x,t)|(x-x_0)|^2\mathrm{d}x.$$

Together with the condition in Theorem 1.2(i), it is easy to find that both $E(0) < 0$ and $E'(s) > 0$ holds for $s > 0$. This yields $E(M_{\Phi(0)}) < 0$ provided $\int_\Omega u_0(x, t)|(x - x_0)|^2 \mathrm{d}x$ small enough. Combining with (4.7) to get

$$E(M_{\Phi(t)}) < E(M_{\Phi(0)}) < 0, \quad t \in (0, T_{\max}). \tag{4.8}$$

We obtain from (4.7) and (4.8) that

$$\begin{aligned} M_{\Phi(t)} &< M_{\Phi(0)} + \int_0^t E(M_{\Phi(s)}) \mathrm{d}s \\ &< M_{\Phi(0)} + \int_0^t E(M_{\Phi(0)}) \mathrm{d}s \\ &= M_{\Phi(0)} + E(M_{\Phi(0)})t. \end{aligned}$$

This concludes that there exists a $T \in (0, \infty)$ such that $M_{\Phi(t)} \to 0$ as $t \to T$. We complete the proof.

Proposition 4.2 *Let $q > r$, $x_0 \in \Omega$ and $0 < r_1 < r_2 < \mathrm{dist}(x_0, \partial\Omega)$ denote the distance between x_0 and $\partial\Omega$. Then there exist $C_{11}, C_{12} > 0$ relying on $r_1, r_2, \mathrm{dist}(x_0, \partial\Omega)$, such that for $t \in (0, T_{\max})$,*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(x, t) \Phi(x - x_0) \mathrm{d}x \\ \le & \left(4 + \frac{\xi\gamma(q - r)|\Omega|}{2\pi q}\right) \int_\Omega u_0(x) \mathrm{d}x - \frac{(\chi\alpha - \xi\gamma|\Omega|^{\frac{q-r}{q}})}{2\pi|\Omega|^{q-1}} \left(\int_\Omega u_0(x) \mathrm{d}x\right)^{q+1} + \\ & C_{11} \int_\Omega u^q(x, t) \mathrm{d}x \left(\int_\Omega u(x, t) \Phi(x) \mathrm{d}x\right) + \\ & C_{12} \int_\Omega u^q(x, t) \mathrm{d}x \left(\int_\Omega u_0(x) \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u(x, t) \Phi(x) \mathrm{d}x\right)^{\frac{1}{2}} + \\ & C_3 \int_\Omega u_0(x) \mathrm{d}x \left(\int_\Omega u^{2q-1}(x, t) \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u(x, t) \Phi(x) \mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned} \tag{4.9}$$

Proof. From (4.2), we have that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u(x, t) \Phi(x) \mathrm{d}x \\ &\le 4 \int_\Omega u_0(x) \mathrm{d}x + \chi \int_\Omega u(x, t) \nabla \Phi(x) \cdot \nabla v(x, t) \mathrm{d}x - \\ &\quad \xi \int_\Omega u(x, t) \nabla \Phi(x) \cdot \nabla w(x, t) \mathrm{d}x \\ &=: 4 \int_\Omega u_0(x) \mathrm{d}x + Y_1 + Y_2. \end{aligned} \tag{4.10}$$

We have from (4.3) that

$$\begin{aligned} Y_1 &:= \chi \int_\Omega u(x, t) \nabla \Phi(x) \cdot \nabla v(x, t) \mathrm{d}x \\ &\le -\frac{\chi\alpha}{2\pi} \int_\Omega u^q(x, t) \mathrm{d}x \int_\Omega u_0(x) \mathrm{d}x + \overline{C}_1 \int_\Omega u^q(x, t) \mathrm{d}x \int_\Omega u(x, t) \Phi(x) \mathrm{d}x + \\ &\quad \overline{C}_2 \int_\Omega u^q(x, t) \mathrm{d}x \left(\int_\Omega u_0(x) \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u(x, t) \Phi(x) \mathrm{d}x\right)^{\frac{1}{2}} + \end{aligned}$$

$$\overline{C}_3 \int_\Omega u_0(x)\mathrm{d}x \left(\int_\Omega u^{2q-1}(x,\,t)\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega u(x,\,t)\,\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}. \tag{4.11}$$

By the Hölder's inequality, one has

$$\int_\Omega u^r(x,\,t)\mathrm{d}x \le \frac{r}{q}\int_\Omega u^q(x,\,t)\mathrm{d}x + \frac{q-r}{q}|\Omega|. \tag{4.12}$$

So, we can obtain from (4.4) and (4.12) that

$$\begin{aligned} Y_2 \le{}& \frac{\xi\gamma r}{2\pi q}\int_\Omega u^q(x,\,t)(x)\mathrm{d}x \int_\Omega u_0(x)\mathrm{d}x + \frac{\xi\gamma(q-r)|\Omega|}{2\pi q}\int_\Omega u_0(x)\mathrm{d}x + \\ & \widetilde{C_1}\left(\frac{r}{q}\int_\Omega u^q(x,\,t)\mathrm{d}x + \frac{q-r}{q}|\Omega|\right)\int_\Omega u(x,\,t)\,\Phi(x)\mathrm{d}x + \\ & \widetilde{C}_2\left(\frac{r}{q}\int_\Omega u^q(x,\,t)\mathrm{d}x + \frac{q-r}{q}|\Omega|\right)\left(\int_\Omega u_0(x)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_\Omega u(x,\,t)\,\Phi(x)\mathrm{d}x\right)^{\frac{1}{2}}. \end{aligned} \tag{4.13}$$

By the similar proof of (4.6), if $\chi\alpha q - \xi\gamma r > 0$, we can obtain that

$$-\frac{\chi\alpha q - \xi\gamma r}{2\pi q|\Omega|^{q-1}}\left(\int_\Omega u(x,\,t)\mathrm{d}x\right)^q \ge -\frac{\chi\alpha q - \xi\gamma r}{2\pi q}\int_\Omega u^q(x,\,t)\mathrm{d}x. \tag{4.14}$$

Combining (4.8), (4.11), (4.13) and (4.14), we complete the proof.

Lemma 4.2 *Let $\Omega \in \mathbf{R}^2$ be a smooth bounded domain and $x_0 \in \Omega$. Assume that $\int_\Omega u_0(x)|x-x_0|^2\mathrm{d}x$ is sufficiently small,* $q > r$, $\chi\alpha q - \xi\gamma r > 0$,

$$\left(\int_\Omega u_0(x)\mathrm{d}x\right)^q > \frac{2\pi q|\Omega|^{q-1}}{\chi\alpha q - \xi\gamma r}\left(4 + \frac{\xi\gamma(q-r)|\Omega|}{2\pi q}\right).$$

Then the solution of (1.4) *blows up in finite time.*

Proof. By the similar proof of Lemma 4.1, we can get the result of Lemma 4.2, and skip the proof for conciseness.

The proof of Theorem 1.2 is completed by Lemmas 4.1 and 4.2.

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