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Exact Solutions to the Bidirectional SK-Ramani Equation

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Abstract: In this paper, the bidirectional SK-Ramani equation is investigated by means of the extended homoclinic test approach and Riemann theta function method, respectively. Based on the Hirota bilinear method, exact solutions including one-soliton wave solution are obtained by using the extended homoclinic approach and one-periodic wave solution is constructed by using the Riemann theta function method. A limiting procedure is presented to analyze in detail the relations between the one periodic wave solution and one-soliton solution.

Key words: Hirota bilinear method; bidirectional Sawada-Kotera equation; extended homoclinic test approach; Riemann theta function

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1 Introduction

In this paper, we focus on the following nonlinear evolution equation (NLEE)

$$5\partial_x^{-1}u_{tt} + 5u_{xxt} - 15uu_t - 15u_x\partial_x^{-1}u_t - 45u^2u_x + 15u_xu_{xx} + 15uu_{3x} - u_{5x} = 0, \tag{1.1}$$

which was formulated in [1] as a bidirectional counterpart of the classical Sawada-Kotera (SK) equation

$$u_t + 45u^2u_x - 15u_xu_{xx} - 15uu_{xxx} + u_{xxxxx} = 0. \tag{1.2}$$

It has been shown that the bilinear form of equation (1.1) is identical to the well-known Ramani equation (see [2]). Therefore, equation (1.1) is also called bSK-Ramani equation (see [3]). In [4], the quasi-multisoliton and bidirectional solitary wave solutions of bSK-Ramani equation (1.1) are obtained by using the Hirota's direct method (see [5]). In [6], its

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periodic solitary wave solutions are obtained by means of Hirota's direct method and ansätz function method. In [7], the Adomian decomposition method is also used to construct the soliton solutions and doubly-periodic solutions of bSK-Ramani equation (1.1).

It is well known that the Hirota's direct method (see [5]) is a powerful tool to construct exact solutions for NLEEs. Once a NLEE is written in bilinear forms, then multi-soliton solutions and rational solutions can be obtained. In [8] and [9], the Hirota's method is extended to directly construct periodic wave solutions of NLEEs with the help of Riemann theta functions. To our knowledge, there is no literature to investigate bSK-Ramani equation (1.1) by using Riemann theta functions. In this paper, based on a more general bilinear form of bSK-Ramani equation (1.1), we give some exact solutions by using the Riemann theta functions method (see [8]–[9]) and extended homoclinic test approach (see [6], [10]–[11]).

The organization of this paper is as follows. In Section 2, some exact solutions of bSK-Ramani equation (1.1) are obtained by using Hirota's direct method and extended homoclinic test approach. In Section 3, one-periodic wave solution of bSK-Ramani equation (1.1) is obtained with the help of Riemann theta function method. Moreover, a limiting procedure is applied to analyze asymptotic behaviour of the one-periodic wave solution, it is proved that the one-periodic wave solution tends to the one-soliton solution. Some conclusions are provided in Section 4.

2 Extended Homoclinic Test Approach to bSK-Ramani Equation (1.1)

In [6], Liu and Dai obtained some exact and periodic solitary wave solutions of bSK-Ramani equation (1.1) with its bilinear form expressed as

$$(5D_t^2 + 5D_x^3D_t - D_x^6)f \cdot f = 0, \tag{2.1}$$

where the bilinear operator D is defined by

$$D_x^nD_t^mf \cdot g = (\partial_x - \partial_{x'})^n(\partial_t - \partial_{t'})^m f(x,t)g(x',t')\Big|_{x'=x,t'=t.} \tag{2.2}$$

By the following transformation

$$u = u_0 - 2\partial_x^2 \ln f, \tag{2.3}$$

where u_0 is a constant solution of bSK-Ramani equation (1.1), then equation (1.1) can be translated into the following bilinear form

$$(5D_t^2 + 5D_x^3 - D_x^6 - 45u_0D_x^2 - 15u_0D_xD_t + 15u_0D_x^4 + c)f \cdot f = 0, \tag{2.4}$$

in which c is an integration constant. Assuming $c = 0$, yields

$$(5D_t^2 + 5D_x^3 - D_x^6 - 45u_0D_x^2 - 15u_0D_xD_t + 15u_0D_x^4)f \cdot f = 0. \tag{2.5}$$

Based on the extended homoclinic test approach (see [6], [10] and [11]), we suppose that the solution of equation (2.5) takes the form

$$f = \mathrm{e}^{-\xi_1} + c_1\cos(\xi_2) + c_2\cosh(\xi_3) + c_3\mathrm{e}^{\xi_1}, \tag{2.6}$$

where $\xi_j = \kappa_jx + \omega_jt + \delta_j$, and κ_j, ω_j, c_j $(j = 1, 2, 3)$ are constants to be determined, δ_j are free constants. Substituting the ansätz equation (2.6) into the bilinear equation (2.5) and equating all the coefficients of different powers of $\mathrm{e}^{-\xi_1}$, e^{ξ_1}, $\cos(\xi_2)$, $\sin(\xi_2)$, $\cosh(\xi_3)$,

$\sinh(\xi_3)$ and constant term to zero, we get a set of algebraic equations for κ_j, ω_j, c_j. By solving this system with the aid of Maple, we get the following results:

Case 1.

$$\begin{cases} c_2 = 0, \\ \kappa_1 = \mathrm{i}\kappa_2, \\ \omega_1 = -\dfrac{1}{20}\mathrm{i}\left(3\sqrt{5}+5\right)\left(3\sqrt{5}u_0 - 15u_0 - 8\kappa_2^2\right)\kappa_2, \\ \omega_2 = -\dfrac{1}{20}\left(3\sqrt{5}+5\right)\left(3\sqrt{5}u_0 - 15u_0 - 8\kappa_2^2\right)\kappa_2. \end{cases} \tag{2.7}$$

By choosing κ_2 as a real number and $c_3 = 1$, f can be written as

$$f = (c_1 + 2)\cos(\xi_1), \tag{2.8}$$

which corresponds to one-periodic solution of bSK-Ramani equation (1.1) as

$$\begin{cases} u = u_0 + \dfrac{2\kappa_2^2}{[\cos(\xi_1)]^2}, \\ \xi_1 = \dfrac{1}{20}\left(3\sqrt{5}+5\right)\left(3\sqrt{5}u_0 t - 15u_0 t - 8\kappa_2^2 t + 5x - 3\sqrt{5}x\right)\kappa_2. \end{cases} \tag{2.9}$$

When $\kappa_2 = \mathrm{i}k_2$, k_2 is a real number and $c_3 = 1$, f can be written as

$$f = (c_1 + 2)\cosh(\xi_1), \tag{2.10}$$

which corresponds to one-soliton solution

$$\begin{cases} u = u_0 - 2k_2^2[\mathrm{sech}(\xi_1)]^2, \\ \xi_1 = \dfrac{1}{20}(3\sqrt{5}+5)(3\sqrt{5}u_0 t - 15u_0 t + 8k_2^2 t + 5x - 3\sqrt{5}x)k_2, \end{cases} \tag{2.11}$$

it allows wave velocity different from the ones

$$u = -2\partial_x^2 \ln(1 + \mathrm{e}^{\eta}) = -\frac{1}{2}\mu^2\left[\mathrm{sech}\left(\frac{\eta}{2}\right)\right]^2 \tag{2.12}$$

with phase variable

$$\eta = \mu x + \left(-\frac{1}{2} \pm \frac{3}{10}\sqrt{5}\right)\mu^3 t + \varphi,$$

in [1].

Case 2.

$$\begin{cases} c_2 = 0, \\ \kappa_1 = \dfrac{1}{2}\sqrt{5u_0}, \\ \kappa_2 = -\dfrac{1}{2}\sqrt{-5u_0}, \\ \omega_1 = -\dfrac{1}{2}\sqrt{5}u_0^{\frac{3}{2}}, \\ \omega_2 = \dfrac{1}{2}\sqrt{5}(-u_0)^{\frac{3}{2}}. \end{cases} \tag{2.13}$$

When $u_0 > 0$, f can be written as

$$f = \mathrm{e}^{\xi_1} + c_1\cosh(\xi_1) + c_3\mathrm{e}^{\xi_1} \qquad \text{with} \quad \xi_1 = \frac{1}{2}\sqrt{5u_0}(u_0 t - x). \tag{2.14}$$

Substituting expression (2.14) into (2.3), we obtain

$$u = u_0 - \frac{5u_0[2c_1c_3 + 4c_3 + c_1^2 + 2c_1]}{2[\mathrm{e}^{\xi_1} + c_1\cosh(\xi_1) + c_3\mathrm{e}^{-\xi_1}]^2}. \tag{2.15}$$

When $c_3 = 1$, we obtain the one-soliton solution

$$u = u_0 - \frac{5}{2}u_0[\mathrm{sech}(\xi_1)]^2. \tag{2.16}$$

When $u_0 < 0$ and $c_3 = 1$, f can be written as

$$f = (c_1 + 2)\cos(\xi_1) \qquad \text{with} \quad \xi_1 = \frac{1}{2}\sqrt{-5u_0}(u_0t - x), \tag{2.17}$$

which corresponds to one-periodic solution

$$u = u_0 - \frac{5u_0}{2[\cos(\xi_1)]^2}. \tag{2.18}$$

In Case 2, it is clear that the amplitude and wave speed are completely determined by the constant solution u_0.

3 One-periodic Wave Solution and Asymptotic Properties to bSK-Ramani Equation (1.1)

3.1 Construction of One-periodic Wave Solution

Nakamura[12]-[13] developed a direct way to construct a kind of quasi-periodic solutions of NLEEs via Hirota bilinear method and Riemann theta function. More recently, this method is extended to investigate NLEEs include both continuous and discrete ones (see [8], [9], [14]–[16]). In the following, we construct the one-periodic wave solution of bSK-Ramani equation (1.1) by this method and discuss its asymptotic property in detail.

In the process of constructing quasi-periodic wave solutions, the nonzero constant c in (2.4) plays an important role. The Riemann theta function of genus N reads

$$\vartheta(\boldsymbol{\xi}) = \vartheta(\boldsymbol{\xi}, \tau) = \sum_{\boldsymbol{n}\in\mathbf{Z}^N} \mathrm{e}^{2\pi\mathrm{i}\langle\boldsymbol{\xi}, \boldsymbol{n}\rangle - \pi\langle\tau\boldsymbol{n}, \boldsymbol{n}\rangle}, \tag{3.1}$$

where the integer vector $\boldsymbol{n} = (n_1, \cdots, n_N)^{\mathrm{T}} \in \mathbf{Z}^N$, and phase variable $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_N)^{\mathrm{T}} \in \mathbf{C}^N$. For two vectors $\boldsymbol{f} = (f_1, \cdots, f_N)^{\mathrm{T}}$ and $\boldsymbol{g} = (g_1, \cdots, g_N)^{\mathrm{T}}$, their inner product is defined by

$$\langle\boldsymbol{f},\ \boldsymbol{g}\rangle = f_1g_1 + f_2g_2 + \cdots + f_Ng_N. \tag{3.2}$$

When $N = 1$, the Riemann theta function (3.1) becomes

$$\vartheta(\boldsymbol{\xi}, \tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}n\boldsymbol{\xi} - \pi n^2\tau}, \tag{3.3}$$

where $\boldsymbol{\xi} = \kappa x + \omega t + \delta$ and $\tau > 0$. To construct one-periodic wave solution, the function $\boldsymbol{f}$ in (2.4) can be taken as

$$\boldsymbol{f} = \vartheta(\boldsymbol{\xi}) = \vartheta(\boldsymbol{\xi}, \tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}n\boldsymbol{\xi} - \pi n^2\tau}. \tag{3.4}$$

It is well known that the Hirota operator D has the following property:

$$G(D_x, D_t)\mathrm{e}^{\xi_1} \cdot \mathrm{e}^{\xi_2} = G(\kappa_1 - \kappa_2, \omega_1 - \omega_2)\mathrm{e}^{\xi_1+\xi_2}, \tag{3.5}$$

where $G(D_x,\, D_t)$ is a polynomial about D_x and D_t. Substituting expression (3.4) into (2.4) with (3.5) leads to

$$\begin{aligned}
&G(D_x,\, D_t)\vartheta(\boldsymbol{\xi},\tau)\cdot\vartheta(\boldsymbol{\xi},\tau)\\
=&\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G(D_x,\, D_t)\mathrm{e}^{2\pi\mathrm{i}n\boldsymbol{\xi}-\pi n^2\tau}\mathrm{e}^{2\pi\mathrm{i}m\boldsymbol{\xi}-\pi m^2\tau}\\
=&\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}G[2\pi\mathrm{i}(n-m)\kappa, 2\pi\mathrm{i}(n-m)\omega]\mathrm{e}^{2\pi\mathrm{i}(n+m)\boldsymbol{\xi}-\pi(n^2+m^2)\tau}\\
\xlongequal{m=m'-n}&\sum_{m'=-\infty}^{\infty}\bar{G}(m')\mathrm{e}^{2\pi\mathrm{i}m'\boldsymbol{\xi}},
\end{aligned}\tag{3.6}$$

where the coefficient of $\mathrm{e}^{2\pi\mathrm{i}m'\boldsymbol{\xi}}$ is denoted as

$$\bar{G}(m')=\sum_{n=-\infty}^{\infty}G(2\pi\mathrm{i}(2n-m')\kappa,\, 2\pi\mathrm{i}(2n-m')\omega)\mathrm{e}^{-\pi[n^2+(n-m')^2]\tau}.\tag{3.7}$$

For each series $\bar{G}(m')$, by shifting summation index by $n'=n-1$, we have

$$\begin{aligned}
\bar{G}(m')=&\sum_{n'=-\infty}^{\infty}G\{2\pi\mathrm{i}[2n'-(m'-2)]\kappa,\, 2\pi i[2n'-(m'-2)]\omega\}\mathrm{e}^{-\pi\tau n'^2+[n'^2+(m'-2)]^2}\cdot\\
&\mathrm{e}^{-2\pi\tau(m'-1)}\\
=&\cdots\\
=&\begin{cases}\bar{G}(0)\mathrm{e}^{-\pi m'^2\tau/2}, & m' \text{ is even;}\\ \bar{G}(1)\mathrm{e}^{-\pi(m'^2-1)\tau/2}, & m' \text{ is odd,}\end{cases}
\end{aligned}\tag{3.8}$$

which implies that $\bar{G}(m')$, $m'\in\mathbf{Z}$ are completely dominated by $\bar{G}(0)$ and $\bar{G}(1)$, that is, if the equations

$$\bar{G}(0)=\bar{G}(1)=0\tag{3.9}$$

are satisfied, then it follows that $\bar{G}(m')=0$, $m'\in\mathbf{Z}$, and the function (3.4) is just the solution for (2.4), namely,

$$G(D_x,\, D_t)\vartheta\cdot\vartheta=0.$$

Combining (3.7) and (3.9) yields

$$\begin{aligned}
\bar{G}(0)=&\sum_{n=-\infty}^{\infty}(4096\pi^6\kappa^6n^6+3840\pi^4\kappa^4n^4u_0+1280\pi^4\kappa^3n^4\omega+720\pi^2\kappa^2n^2u_0^2+\\
&240\pi^2\kappa n^2\omega u_0-80\pi^2n^2\omega^2+c)\mathrm{e}^{-2\pi n^2\tau}\\
=&\,0,
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
\bar{G}(1)=&\sum_{n=-\infty}^{\infty}[64\pi^6\kappa^6(2n-1)^6+240\pi^4\kappa^4(2n-1)^4u_0+80\pi^4\kappa^3(2n-1)^4\omega+\\
&180\pi^2\kappa^2(2n-1)^2u_0^2+60\pi^2\kappa(2n-1)^2\omega u_0-20\pi^2(2n-1)^2\omega^2+c]\cdot\\
&\mathrm{e}^{-\pi(2n^2-2n+1)\tau}\\
=&\,0.
\end{aligned}\tag{3.11}$$

By introducing the notations as

$$\begin{cases} \lambda = \mathrm{e}^{-\pi\tau}, \\ \eta = \omega^2, \\ a_{11} = \sum_{n=-\infty}^{\infty} (1280\pi^4\kappa^3 n^4 + 240\pi^2\kappa n^2 u_0)\lambda^{2n^2}, \\ a_{12} = \sum_{n=-\infty}^{\infty} 80\pi^2 n^2 \lambda^{2n^2}, \\ a_{13} = \sum_{n=-\infty}^{\infty} \lambda^{2n^2}, \\ a_{21} = \sum_{n=-\infty}^{\infty} (80\pi^4\kappa^3(2n-1)^4 + 60\pi^2\kappa(2n-1)^2 u_0)\lambda^{2n^2-2n+1}, \\ a_{22} = -\sum_{n=-\infty}^{\infty} 20\pi^2(2n-1)^2\lambda^{2n^2-2n+1}, \\ a_{23} = \sum_{n=-\infty}^{\infty} \lambda^{2n^2-2n+1}, \\ b_1 = -\sum_{n=-\infty}^{\infty} (4096\pi^6\kappa^6 n^6 + 3840\pi^4\kappa^4 n^4 u_0 + 720\pi^2\kappa^2 n^2 u_0^2)\lambda^{2n^2}, \\ b_2 = -\sum_{n=-\infty}^{\infty} [64\pi^6\kappa^6(2n-1)^6 + 240\pi^4\kappa^4(2n-1)^4 u_0 + 180\pi^2\kappa^2(2n-1)^2 u_0^2]\cdot \\ \qquad \lambda^{2n^2-2n+1}, \end{cases} \tag{3.12}$$

then system (3.10) and (3.11) are simplified into a linear system for the frequency ω and the integration constant c and η, namely

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} \omega \\ \eta \\ c \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}. \tag{3.13}$$

There are only two constraint equations with three constants ω, c and η in system (3.13), however, thanks to the third constraint equation

$$\eta = \omega^2, \tag{3.14}$$

we have

$$\omega = \frac{-B' \pm \sqrt{B'^2 - 4A'C'}}{2A'}, \tag{3.15}$$

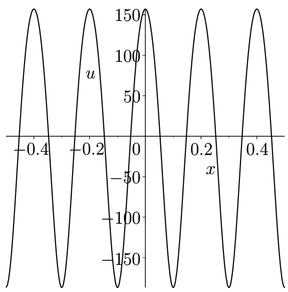
with

$$A' = a_{22} - \frac{a_{12}a_{23}}{a_{13}}, \quad B' = a_{21} - \frac{a_{11}a_{23}}{a_{13}}, \quad C' = \frac{a_{23}}{a_{13}}b_1 - b_2. \tag{3.16}$$

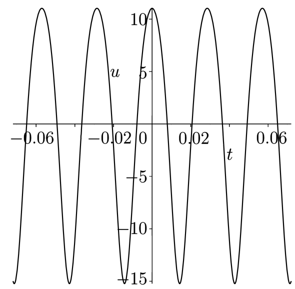
Now we obtain a one-periodic wave solution of bSK-Ramani equation (1.1)

$$u = u_0 - 2\partial_x^2 \ln\vartheta(\boldsymbol{\xi}), \tag{3.17}$$

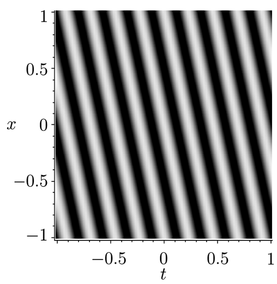
where $\vartheta(\boldsymbol{\xi})$ and ω are given by (3.4) and (3.15), respectively. The other parameters κ, δ, τ, u_0 are free. Fig. 3.1 shows the one-periodic wave (3.17) for proper choices of parameters.



(a) The wave propagation pattern of the wave along the x axis with the parameters $u_0 = 0$, $\delta = 0$, $\kappa = 5$, $\tau = 1$



(b) The wave propagation pattern of the wave along the t axis with the parameters $u_0 = 0$, $\delta = 0$, $\kappa = 1$, $\tau = 0.8$



(c) The parameters are $u_0 = 0$, $\delta = 0$, $\kappa = 1$, $\tau = 1$. Overhead view of the wave, with contour plot shown. The bright lines are crests and the dark lines are troughs

Fig. 3.1 A one periodic wave of bSK-Ramani equation (1.1) via expression (3.17)

3.2 Asymptotic Property of One-periodic Wave Solution (3.17)

In the following, we consider asymptotic properties of the one-periodic wave solution (3.17). Since both the coefficient matrix and the right-side vector of system (3.13) are power series of λ, its solution $(\omega, \eta, c)^{\mathrm{T}}$ can be written as a power series of λ. We can solve system (3.13) through small parameter expansion method, the general procedure is described as follow.

We write the system (3.13) into power series of λ:

$$\begin{cases} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \boldsymbol{A}_0 + \boldsymbol{A}_1\lambda + \boldsymbol{A}_2\lambda^2 + \cdots, \\ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \boldsymbol{B}_0 + \boldsymbol{B}_1\lambda + \boldsymbol{B}_2\lambda^2 + \cdots, \\ \begin{pmatrix} \omega \\ \eta \\ c \end{pmatrix} = \boldsymbol{X}_0 + \boldsymbol{X}_1\lambda + \boldsymbol{X}_2\lambda^2 + \cdots, \end{cases} \tag{3.18}$$

where

$$\boldsymbol{A}_k = \begin{pmatrix} a_{11}^{(k)} & a_{12}^{(k)} & a_{13}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} & a_{23}^{(k)} \end{pmatrix}, \quad \boldsymbol{B}_k = \begin{pmatrix} b_1^{(k)} \\ b_2^{(k)} \end{pmatrix}, \quad \boldsymbol{X}_k = \begin{pmatrix} \omega^{(k)} \\ \eta^{(k)} \\ c^{(k)} \end{pmatrix}, \qquad k = 0, 1, \cdots \tag{3.19}$$

with $a_{11}^{(k)}$, $a_{12}^{(k)}$, $a_{13}^{(k)}$, $a_{21}^{(k)}$, $a_{22}^{(k)}$, $a_{23}^{(k)}$, $b_1^{(k)}$, $b_2^{(k)}$, $\omega^{(k)}$, $\eta^{(k)}$, $c^{(k)}$ are coefficients of the kth power of λ in (3.12). Substituting (3.18) into (3.13) derives the following recursion relations

$$\begin{cases} \boldsymbol{A}_0\boldsymbol{X}_0 = \boldsymbol{B}_0, \\ \boldsymbol{A}_0\boldsymbol{X}_1 + \boldsymbol{A}_1\boldsymbol{X}_0 = \boldsymbol{B}_1, \\ \boldsymbol{A}_0\boldsymbol{X}_2 + \boldsymbol{A}_1\boldsymbol{X}_1 + \boldsymbol{A}_2\boldsymbol{X}_0 = \boldsymbol{B}_2, \\ \qquad\qquad \vdots \\ \boldsymbol{A}_0\boldsymbol{X}_k + \boldsymbol{A}_1\boldsymbol{X}_{k-1} + \cdots + \boldsymbol{A}_k\boldsymbol{X}_0 = \boldsymbol{B}_k, \end{cases} \tag{3.20}$$

from which we can recursively get each vector $\boldsymbol{X}_k$, $k = 0, 1, \cdots$.

If the matrix $\boldsymbol{A}_0$ is reversible, solving relations (3.20) yields

$$\boldsymbol{X}_0 = \boldsymbol{A}_0^{-1}\boldsymbol{B}_0, \quad \boldsymbol{X}_k = \boldsymbol{A}_0^{-1}\left(\boldsymbol{B}_k - \sum_{j=1}^{k}\boldsymbol{A}_j\boldsymbol{B}_{k-1}\right), \qquad k = 1, 2, \cdots .$$

If $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$ are not reversible, but they take the following form

$$\boldsymbol{A}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} \end{pmatrix}, \tag{3.21}$$

considering the third constraint equation

$$\eta = \omega^2 = \eta^{(0)} + \eta^{(1)}\lambda + \eta^{(2)}\lambda^2 + \cdots = (\omega^{(0)} + \omega^{(1)}\lambda + \omega^{(2)}\lambda^2 + \cdots)^2, \tag{3.22}$$

and comparing the coefficient of same power of λ in (3.22), we get

$$\eta^{(k)} = \sum_{j=0}^{k}\omega^{(j)}\omega^{(k-j)}. \tag{3.23}$$

Solving relations (3.20) with (3.21) and (3.23) yields

$$c^{(0)} = \boldsymbol{B}_0^{(\mathrm{I})}, \qquad \omega^{(0)} = \frac{-a_{21}^{(1)} \pm \sqrt{[a_{21}^{(1)}]^2 + 4a_{22}^{(1)}b_2^{(1)}}}{2a_{22}^{(1)}},$$

$$\boldsymbol{X}_0 = \begin{pmatrix} \omega^{(0)} \\ \eta^{(0)} \\ c^{(0)} \end{pmatrix} = \begin{pmatrix} \omega^{(0)} \\ [\omega^{(0)}]^2 \\ \boldsymbol{B}_0^{(\mathrm{I})} \end{pmatrix},$$

$$\boldsymbol{X}_1 = \begin{pmatrix} \omega^{(1)} \\ \eta^{(1)} \\ c^{(1)} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{a_{21}^{(1)} + 2a_{22}^{(1)}\omega^{(0)}}[(\boldsymbol{B}_2 - \boldsymbol{A}_2\boldsymbol{X}_0)^{(\mathrm{II})} - a_{23}^{(1)}\boldsymbol{B}_1^{(\mathrm{I})}] \\ \dfrac{2\omega^{(0)}}{a_{21}^{(1)} + 2a_{22}^{(1)}\omega^{(0)}}[(\boldsymbol{B}_2 - \boldsymbol{A}_2\boldsymbol{X}_0)^{(\mathrm{II})} - a_{23}^{(1)}\boldsymbol{B}_1^{(\mathrm{I})}] \\ \boldsymbol{B}_1^{(\mathrm{II})} \end{pmatrix},$$

$$\boldsymbol{X}_k = \begin{pmatrix} \omega^{(k)} \\ \eta^{(k)} \\ c^{(k)} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{a_{21}^{(1)} + 2a_{22}^{(1)}\omega^{(0)}}\left(\boldsymbol{P}^{(\mathrm{II})} - a_{23}^{(1)}\boldsymbol{Q}^{(\mathrm{I})} - a_{22}^{(1)}\displaystyle\sum_{j=1}^{k-1}\omega^{(i)}\omega^{(k-i)}\right) \\ \displaystyle\sum_{j=0}^{k}\omega^{(j)}\omega^{(k-j)} \\ \boldsymbol{Q}^{(\mathrm{I})} \end{pmatrix},$$

$$k = 2, 3, \cdots , \tag{3.24}$$

where $\boldsymbol{V}^{(\mathrm{I})}$ and $\boldsymbol{V}^{(\mathrm{II})}$ denote the the first and second component of a two-dimensional vector $\boldsymbol{V}$ respectively, and

$$\boldsymbol{P} = \boldsymbol{B}_{k+1} - \sum_{j=2}^{k+1} \boldsymbol{A}_j \boldsymbol{X}_{k+1-j},$$

$$\boldsymbol{Q} = \boldsymbol{B}_{k+1} - \sum_{j=2}^{k} \boldsymbol{A}_j \boldsymbol{X}_{k-j}.$$

The relation between the one-periodic wave solution (3.17) and the one-soliton solution (2.12) can be established as follows.

With the help of (3.12) with (3.18), we write a_{ij}, b_i, $i = 1, 2$, $j = 1, 2, 3$ as the series of λ

$$\begin{cases} a_{11} = (2560\pi^4\kappa^3 + 480\pi^2\kappa u_0)\lambda^2 + (40960\pi^4\kappa^3 + 1920\pi^2\kappa u_0)\lambda^8 \cdots , \\ a_{12} = -160\pi^2\lambda^2 - 640\pi^2\lambda^8 \cdots , \\ a_{13} = 1 + 2\lambda^2 + 2\lambda^8 + \cdots , \\ a_{21} = (160\pi^4\kappa^3 + 120\pi^2\kappa u_0)\lambda + (6560\pi^4\kappa^3 + 600\pi^2\kappa u_0)\lambda^5 + \cdots , \\ a_{22} = -40\pi^2\lambda - 200\pi^2\lambda^5 + \cdots , \\ a_{23} = 2\lambda + 2\lambda^5 + \cdots , \\ b_1 = (-8192\pi^6\kappa^6 - 7680\pi^4\kappa^4 u_0 - 1440\pi^2\kappa^2 u_0^2)\lambda^2 + \\ \qquad (-524288\pi^6\kappa^6 - 122880\pi^4\kappa^4 u_0 - 5760\pi^2\kappa^2 u_0^2)\lambda^8 + \cdots , \\ b_2 = (-128\pi^6\kappa^6 - 480\pi^4\kappa^4 u_0 - 360\pi^2\kappa^2 u_0^2)\lambda + \\ \qquad (-93312\pi^6\kappa^6 - 38880\pi^4\kappa^4 u_0 - 3240\pi^2\kappa^2 u_0^2)\lambda^5 + \cdots , \end{cases} \tag{3.25}$$

then, we have

$$\begin{cases} \boldsymbol{A}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ \boldsymbol{A}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 160\pi^4\kappa^3 + 120\pi^2\kappa u_0 & -40\pi^2 & 2 \end{pmatrix}, \\ \boldsymbol{A}_2 = \begin{pmatrix} 2560\pi^4\kappa^3 + 480\pi^2\kappa u_0 & -160\pi^2 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \\ \boldsymbol{A}_3 = 0, \\ \boldsymbol{A}_4 = 0, \\ \boldsymbol{A}_5 = \begin{pmatrix} 0 & 0 & 0 \\ 6560\pi^4\kappa^3 + 600\pi^2\kappa u_0 & -200\pi^2 & 2 \end{pmatrix}, \\ \qquad \vdots \end{cases} \tag{3.26}$$

and

$$\begin{cases} \boldsymbol{B}_0 = 0, \\ \boldsymbol{B}_1 = \begin{pmatrix} 0 \\ -128\pi^6\kappa^6 - 480\pi^4\kappa^4 u_0 - 360\pi^2\kappa^2 u_0^2 \end{pmatrix}, \\ \boldsymbol{B}_2 = \begin{pmatrix} -8192\pi^6\kappa^6 - 7680\pi^4\kappa^4 u_0 - 1440\pi^2\kappa^2 u_0^2 \\ 0 \end{pmatrix}, \\ \boldsymbol{B}_3 = 0, \\ \boldsymbol{B}_4 = 0, \\ \boldsymbol{B}_5 = \begin{pmatrix} 0 \\ -93312\pi^6\kappa^6 - 38880\pi^4\kappa^4 u_0 - 3240\pi^2\kappa^2 u_0^2 \end{pmatrix}, \\ \qquad \vdots \end{cases} \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.24), we get $\boldsymbol{X}_k$ $(k = 0, 1, 2, \cdots)$, and thus

$$\begin{aligned} c = \Bigg[& -8192\pi^6\kappa^6 - 7680\pi^4\kappa^4 u_0 - 1440\pi^2\kappa^2 u_0^2 - \kappa\left(2560\pi^4\kappa^3 + 480\pi^2\kappa u_0\right) \\ & \left(2\pi^2\kappa^2 + \frac{3}{2}u_0 + \frac{3}{10}\sqrt{80\pi^4\kappa^4 + 200\pi^2\kappa^2 u_0 + 125u_0^2}\right) + \\ & 160\pi^2\kappa^2\left(2\pi^2\kappa^2 + \frac{3}{2}u_0 + \frac{3}{10}\sqrt{80\pi^4\kappa^4 + 200\pi^2\kappa^2 u_0 + 125u_0^2}\right)\Bigg]\lambda^2 + o(\lambda^2), \\ \omega = \kappa & \left(2\pi^2\kappa^2 + \frac{3}{2}u_0 \pm \frac{3}{10}\sqrt{80\pi^4\kappa^4 + 200\pi^2\kappa^2 u_0 + 125u_0^2}\right) - \\ & \frac{48\pi^2\kappa^3\left(20\pi^2\kappa^2 + 15u_0 + \sqrt{80\pi^4\kappa^4 + 200\pi^2\kappa^2 u_0 + 125u_0^2}\right)}{\sqrt{80\pi^4\kappa^4 + 200\pi^2\kappa^2 u_0 + 125u_0^2}}\lambda^2 + o(\lambda^2), \end{aligned}$$

which implies that by setting $u_0 = 0$

$$c \to 0, \quad 2\pi\mathrm{i}\omega \to -8\pi^3\mathrm{i}\kappa^3\left(-\frac{1}{2} \pm \frac{3}{10}\sqrt{5}\right) \qquad \text{as } \lambda \to 0. \tag{3.28}$$

To show that the one-periodic wave (3.17) tends to the one-soliton solution (2.12), under the transformation

$$\hat{\xi} = 2\pi\mathrm{i}\xi - \pi\tau = \mu x + 2\pi\mathrm{i}\omega t + \varphi, \tag{3.29}$$

the theta function (3.4)

$$\vartheta(\xi, \tau) = 1 + \lambda(\mathrm{e}^{2\pi\mathrm{i}\xi} + \mathrm{e}^{-2\pi\mathrm{i}\xi}) + \lambda^4(\mathrm{e}^{4\pi\mathrm{i}\xi} + \mathrm{e}^{-4\pi\mathrm{i}\xi}) + \cdots \tag{3.30}$$

can be written as

$$\vartheta(\xi, \tau) = 1 + \mathrm{e}^{\hat{\xi}} + \lambda^2(\mathrm{e}^{-\hat{\xi}} + \mathrm{e}^{2\hat{\xi}}) + \lambda^6(\mathrm{e}^{-2\hat{\xi}} + \mathrm{e}^{3\hat{\xi}}) + \cdots \to 1 + \mathrm{e}^{\hat{\xi}} \qquad \text{as } \lambda \to 0. \tag{3.31}$$

Combing (3.28) and (3.29) deduces that

$$\hat{\xi} \to \mu x + \left(-\frac{1}{2} \pm \frac{3}{10}\sqrt{5}\right)\mu^3 t + \varphi = \eta \qquad \text{as } \lambda \to 0 \tag{3.32}$$

or, equivalently,

$$\xi = \frac{\eta + \pi\tau}{2\pi\mathrm{i}} \qquad as\ \lambda \to 0. \tag{3.33}$$

(3.32) immediately leads to

$$\vartheta(\xi,\tau)\to 1+\mathrm{e}^{\eta}\qquad \text{as } \lambda\to 0.$$

From above, we conclude that the one-periodic wave (3.4) tends to the one-soliton solution (2.12) as the amplitude $\lambda\to 0$.

4 Conclusions

The investigation on various exact solutions of NLEEs is very important in nonlinear science (see [17]). In this paper, based on the bilinear equation (2.4), the extended homoclinic test approach and Riemann theta function method are applied to construct exact solutions of the bSK-Ramani equation (1.1), respectively. With the help of the Riemann theta function and Hirota bilinear method, the one periodic wave solution (3.17) of the bSK-Ramani equation (1.1) is obtained. A limiting procedure is presented to analyze asymptotic behaviour of the one-periodic wave, it is proved that the one-periodic wave solution (3.17) tends to the one-soliton solution (2.12) under small amplitude limit. It should be noted that the system (3.13) is different from the general form in [8]–[9], [14]–[16] and [18]–[19]. However, it is difficult to construct the two-periodic wave solution of the bSK-Ramani equation (1.1), the reason is that the term D_t^2 in bilinear equation (2.4). How to construct two-periodic wave solution for this type of bilinear equation by Riemann theta function method is worthy of further study.

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