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The Transfer Ideal under the Action of a Nonmetacyclic Group in the Modular Case

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Abstract: Let F_q be a finite field of characteristic p ($p \neq 2$) and V_4 a four-dimensional F_q-vector space. In this paper, we mainly determine the structure of the transfer ideal for the ring of polynomials $F_q[V_4]$ under the action of a nonmetacyclic p-group P in the modular case. We prove that the height of the transfer ideal is 1 using the fixed point sets of the elements of order p in P and that the transfer ideal is a principal ideal.

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1 Introduction

Let V be a vector space of dimension n over a field F and we choose a basis, $\{x_1, x_2, \cdots, x_n\}$, for the dual, V^*, of V. Let $\rho: G \hookrightarrow GL(n, F)$ be a faithful representation of a finite group G. The action of G on V induces an action on V^* which extends to an action by algebra automorphisms on the symmetric algebra of V^*,

$$F[V] = F[x_1, x_2, \cdots, x_n].$$

Specially, for every $g \in G$, $f \in F[V]$ and $\boldsymbol{v} \in V$,

$$(g \cdot f)(\boldsymbol{v}) = f(\rho(g^{-1})\boldsymbol{v}).$$

The ring of polynomial invariants of G is the subring of $F[V]$ given by

$$F[V]^G := \{f \in F[V] \mid gf = f, \ g \in G\}.$$

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The transfer homomorphism is defined by

$$\begin{aligned}\mathrm{Tr}^G\colon F[V] &\to F[V]^G,\\ f &\mapsto \sum_{g\in G} gf\end{aligned}$$

and it is a homomorphism of $F[V]^G$-modules. It is surjective in the nonmodular case, where it provides a tool for constructing the ring of polynomial invariants $F[V]^G$. In the modular case, the image of the transfer $\mathrm{Im}(\mathrm{Tr}^G) \subseteq F[V]^G$ is a proper, non-zero ideal[1]. This makes the transfer ideal an interesting object of study and inspired quite a number of research. Campbell[2] proved that the transfer ideal is a principal ideal for $GL_n(F_q)$ and $U_n(F_q)$ using block bases, where F_q denotes the finite field. He also proved that the transfer ideal is in general not principal for Σ_n. Neusel[3],[4] discussed the transfer for permutation representations. She proved that the transfer ideal for cyclic p-groups is a prime ideal of height at most $n-k$, where k denotes the number of orbits of a permutation basis. She also showed that the transfer ideal for permutation representations of finite groups is generated by the transfers of special monomials. And this leads to a description of the transfer ideal for the alternating groups. In this paper, we discuss the transfer ideal for a nonmetacyclic p-group.

The group P defined by generators a, b, c and the relations

$$\begin{cases} a^p = b^p = c^p = e \qquad (p \neq 2),\\ [a,\, b] = aba^{-1}b^{-1} = c,\\ [a,\, c] = [b,\, c] = e \end{cases}$$

is a nonmetacyclic p-group. It is easy to see that

$$P = \{a^i b^j c^k \mid 0 \leq i,\, j,\, k \leq p-1\},$$

and the order of every element is p except the identity element, whence $|P| = p^3$. The p-group P is the nonabelian p-group of smallest order (see [5]). In Section 2, we introduce a faithful representation $\phi\colon P \hookrightarrow GL(4,\, F_q)$ of P over the field F_q. Here F_q denotes the field with q elements and the characteristic of F_q is p ($p \neq 2$). For the representation of finite p-groups in characteristic p, there is always a Dade basis (see [1]). So we can construct a system of parameters for the ring of polynomial invariants $F_q[V_4]^P$ by finding a Dade basis. Moreover, the product of the degrees of the system of parameters is equal to the order of P, so the system of parameters can generate $F_q[V_4]^P$. Then we calculate the ring of coinvariants $F_q[V_4]_P$ and show that it is not the regular representation. In Section 3, we first calculate the fixed point sets of the elements of order p in group P and obtain the height of the transfer ideal is 1, i.e., $\mathrm{ht}(\mathrm{Im}(\mathrm{Tr}^P)) = 1$. The image of a set of module generators for $F[V]$ as a $F[V]^G$-module under the transfer map can generate the transfer ideal. We show that $F_q[V_4]$ is a free $F_q[V_4]^P$-module. Furthermore, we use this result to prove that the transfer ideal $\mathrm{Im}(\mathrm{Tr}^P)$ is a principal ideal.

2 The Ring of Invariants and the Ring of Coinvariants

First, we introduce a faithful representation $\phi\colon P \hookrightarrow GL(4,\, F_q)$ of P over the field F_q.