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On the Existence of Time-periodic Solution to the Compressible Heat-conducting Navier-Stokes Equations

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Abstract: We study in this article the compressible heat-conducting Navier-Stokes equations in periodic domain driven by a time-periodic external force. The existence of the strong time-periodic solution is established by a new approach. First, we reformulate the system and consider some decay estimates of the linearized system. Under some smallness and symmetry assumptions on the external force, the existence of the time-periodic solution of the linearized system is then identified as the fixed point of a Poincaré map which is obtained by the Tychonoff fixed point theorem. Although the Tychonoff fixed point theorem cannot directly ensure the uniqueness, but we could construct a set-valued function, the fixed point of which is the time-periodic solution of the original system. At last, the existence of the fixed point is obtained by the Kakutani fixed point theorem. In addition, the uniqueness of time-periodic solution is also studied.

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1 Introduction

In this paper, we prove the existence and uniqueness of the strong time-periodic solution to the Navier-Stokes equations for compressible heat-conducting fluids:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

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$$\rho(u_t + (u\cdot\nabla)u) + \nabla P(\rho,\, \theta) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u + \rho f(x,\, t), \tag{1.2}$$

$$\rho C_\nu(\theta_t + u\cdot\nabla\theta) + \theta P_\theta(\rho,\, \theta)\mathrm{div} u = \kappa\Delta\theta + \Phi(u), \tag{1.3}$$

when the external force is time-periodic with T-period

$$f(t+T,\, x) = f(t,\, x)$$

for all t, x. Here, $\rho(x,\, t)$, $u(x,\, t) = (u_1,\, \cdots,\, u_n)(x,\, t)$, $\theta(x,\, t)$ represent the fluid density, velocity and temperature, respectively. $t \in \mathbf{R}$ is the time and x is the spatial variable confined to $\Omega \subset \mathbf{R}^n$ with $n \geq 3$. $P(\rho,\, \theta)$ is the pressure which is a smooth function of ρ, θ. μ, λ are the viscosity coefficients which are assumed to satisfy the physical restrictions

$$\mu > 0, \qquad \frac{n}{2}\lambda + \mu \geq 0.$$

The constants C_ν and κ are the heat capacity at constant volume and the coefficient of heat conductivity. The classical dissipation function $\Phi(u)$ is given by

$$\Phi(u) = \frac{\mu}{2}\sum_{i,j=1}^{n}(\partial_i u_j + \partial_j u_i)^2 + \lambda\sum_{j=1}^{n}(\partial_j u_j)^2.$$

Throughout the paper, we consider $\Omega := [-L,\, L]^n$. Let the density and the temperature satisfy the obvious physical requirements

$$\int_\Omega \rho \mathrm{d}x = \bar\rho > 0, \qquad \theta|_{\partial\Omega} = \bar\theta, \tag{1.4}$$

where $\bar\rho$ and $\bar\theta$ are given constants. Assume that the external force

$$f(x,\, t) = (f_1,\, \cdots,\, f_n)(x,\, t)$$

is spacial periodic with period $2L$ and satisfies

$$f_i(Y_i(x),\, t) = -f_i(x,\, t), \quad f_i(Y_j(x),\, t) = f_i(x,\, t), \qquad \forall\, i \neq j, \tag{1.5}$$

for all $i = 1,\, \cdots,\, n$ and $t \in \mathbf{R}$, where

$$Y_i[x_1,\, \cdots,\, x_i,\, \cdots, x_n] = [x_1,\, \cdots,\, -x_i,\, \cdots,\, x_n].$$

These conditions are to ensure that the Poincaré inequality holds. In fact, we can consider the following no-stick boundary conditions for the velocity:

$$u(t,\, x)\cdot n(x) = 0, \quad [\mathrm{D}u(t,\, x)\cdot n(x)]_\tau = 0 \qquad \text{on } \partial\Omega', \tag{1.6}$$

where $n(x)$ denotes the outer normal vector and $[w(x,\, t)]_\tau$ is the projection of a vector $w(t,\, x)$ on the tangent plane to $\partial\Omega'$ at the point x. In the n-dimensional case and the boundary is flat, (1.6) means that the vorticity is perpendicular to the boundary. For the physical background as well as further properties of flows on domains with frictionless boundary, we refer to [1]. To simplify the presentation, we restrict our attention to a particular class of spatial domains, specifically, we assume that Ω' is an n-dimensional cube:

$$\Omega' = [0,\, L]^n.$$

Then, the boundary conditions (1.6) read as

$$u_i = 0$$

on the opposite faces

$$\{x_i = 0,\, x_j = [0,\, L],\, i \neq j\}\bigcup\{x_i = L,\, x_j = [0,\, L],\, i \neq j\},$$

$$\frac{\partial u_j}{\partial x_i} = 0$$