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Normal Families of Holomorphic Functions Concerning Zero Numbers

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Abstract: In this paper, we study the normal families related with a Hayman conjecture of higher derivative concerning zero numbers, and get one normal criteria. Our result improve some earlier related result.

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1 Introduction and Main Results

Let $\mathcal{F}$ be a meromorphic function in $\mathbf{C}$, and D be a domain in $\mathbf{C}$. $\mathcal{F}$ is said to be normal in D if any sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence f_{n_j} such that f_{n_j} converges spherically locally uniformly in D, to a meromorphic function or ∞ (see [1]–[3]).

In 1959, Hayman[4] proved the following result.

Theorem 1.1 *Let f be a meromorphic function in $\mathbf{C}$, $n \geq 5$ be a positive integer, and $a(\neq 0)$, b be two finite constants. If $f' - af^n \neq b$, then f is a constant.*

The following normality criterion corresponding to Hayman's result was proved by Drasin[5] and Ye[6].

Theorem 1.2 *Let $n \geq 2$ be a positive integer, $a(\neq 0)$, b be two finite constants, and $\mathcal{F}$ be a family of Holomorphic functions in a domain D. If for each $f \in \mathcal{F}$, $f' - af^n \neq b$, then $\mathcal{F}$ is normal in D.*

Recently, by the idea of concerning zero numbers, Deng *et al.*[7] proved the following result.

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Theorem 1.3 *Let m, n, k be three positive integers satisfying $n \geq m+1$, $a(\neq 0)$, b be two finite constants, and $\mathcal{F}$ be a family of Holomorphic functions in a domain D, all of whose zeros have multiplicity at least k. If for each function $f \in \mathcal{F}$, $f^{(k)} - af^n - b$ has at most mk distinct zeros in D, then $\mathcal{F}$ is normal in D.*

A natural problem arises: what can we say if $f^{(k)}$ in Theorem 1.3 is replaced by the $(f^{(k)})^d$? In this paper, we prove the following result.

Theorem 1.4 *Let m, n, k, d be four positive integers satisfying $n \geq (m+1)d$, $a(\neq 0)$, b be two finite constants, and $\mathcal{F}$ be a family of holomorphic functions in a domain D, all of whose zeros have multiplicity at least k. If for each function $f \in \mathcal{F}$, $(f^{(k)})^d - af^n - b$ has at most mdk distinct zeros in D, then $\mathcal{F}$ is normal in D.*

Example 1.1 Let n, k, d be three positive integers, a be a nonzero finite constant, and $\mathcal{F} = \{f_j = jz^{k-1} \colon j = 1, 2, 3, \cdots\}$, $D = \{z \colon |z| < 1\}$. Then, for each $f \in \mathcal{F}$, $(f^{(k)})^d - af^n - 0$ has just one distinct zero in D, but $\mathcal{F}$ is not normal in D. This shows that the zeros of function $f \in \mathcal{F}$ have multiplicity at least k is necessary in Theorem 1.4.

Example 1.2 Let n, k, d be three positive integers, a be a nonzero finite constant, and $\mathcal{F}$=$\{f_j = jz^k \colon j = 1, 2, 3, \cdots\}$, $D = \{z \colon |z| < 1\}$. Then, for each $f \in \mathcal{F}$, $(f^{(k)})^d - af^{(m+1)d-1} - 0$ has exactly $[(m+1)d-1]k \geq mdk$ distinct zero in D, and $(f^{(k)})^d - af^{(m+1)d} - 0$ has exactly $(m+1)dk \geq mdk + 1$, but $\mathcal{F}$ is not normal in D. This shows that both $n \geq (m+1)d$ and $(f^{(k)})^d - af^n - b$ have at most mdk distinct zeros in Theorem 1.4 are best possible.

2 Some Lemmas

In order to prove our theorems, we require the following results.

Lemma 2.1[8] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc Δ satisfying all zeros of functions in $\mathcal{F}$ have multiplicity $\geq p$ and all poles of functions in $\mathcal{F}$ have multiplicity $\geq q$. Let α be a real number satisfying $-p < \alpha < q$. Then $\mathcal{F}$ is not normal at a point z_0 if and only if there exist*

(i) *points $z_n \in \Delta$, $z_n \to z_0$;*

(ii) *positive numbers ρ_n, $\rho_n \to 0$;*

(iii) *functions $f_n \in \mathcal{F}$*

such that $\rho_n^\alpha f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically uniformly on each compact subset of $\mathbf{C}$, where $g(\zeta)$ is a nonconstant meromorphic function satisfying the zeros of g are of multiplicities $\geq p$ and the poles of g are of multiplicities $\geq q$. Moreover, the order of g is at most 2. If g is holomorphic, then g is of exponential type and the order of g is at most 1.

Lemma 2.2[9] *Let f be a nonconstant meromorphic (entire) function in the complex plane, $a(\neq 0)$ be a finite constant, and n be a positive integer with $n \geq 4$ ($n \geq 2$). Then $f' - af^n$ has at least two distinct zeros.*