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Theorems of Helly's Type for Partially-closed Half-spaces

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Abstract: In this article, under quite weaker conditions, theorems of Helly's type for (not necessarily finite) family of partially closed half-spaces are presented in both analytic and geometric forms. Examples are also provided to show that these conditions cannot be omitted in general.

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1 Introduction

The well-known classical Helly's Theorem (see [1]), which states that all members in a finite family of convex sets in the Euclidean space $\mathbf{R}^n$ have a common point if arbitrary $n+1$ members in the family have a common point, is very profound and important in convex geometry and combinatorial geometry. Also it turns out that Helly's Theorem has various elegant applications in many branches of mathematics (see [2]–[4]). For its importance, Helly's theorem has gained much attention since it was found. Various kinds of extensions or generalizations of Helly's Theorem were proved and the corresponding applications were established in the past years (see [5]–[8]). We refer readers to Danzer's seminal paper [9] for general references.

There are examples showing that the finiteness of the family in Helly's Theorem cannot be omitted in general. So, among the extensions or generalizations of Helly's Theorem, one attempt is to find the conditions (for convex sets) under which Helly's Theorem holds for infinite families. For instance, Helly's Theorem holds for (not necessarily finite) families of

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compact convex sets (see [10] and [11]).

Theorems of Helly's type for half-spaces with particular conditions appear often in many branches of mathematics. For instance, it was shown in [10] that Helly's Theorem holds for families of closed half-spaces with the condition that the intersection of some finitely many members in the family is compact, from which the existence of center for Borel probabilities is confirmed, and also Theorem 3.1 in [12] (see also [13]) is in principle a theorem of Helly's type in analytic form for families of closed half-spaces with another condition. In this paper, we present, under quite weaker conditions, theorems of Helly's type for (not necessarily finite) families of partially closed half-spaces (see below for definition) in both analytic and geometric forms, along with examples showing that these conditions cannot be omitted in general.

2 Notation and Definitions

We work with the Euclidean space $\mathbf{R}^n$ and do not distinguish points and vectors in $\mathbf{R}^n$ intentionally. Let $\mathcal{K}^n$ denote the family of all convex bodies (i.e., compact convex sets with nonempty interior) in $\mathbf{R}^n$. For any subset $S \subset \mathbf{R}^n$, $\mathrm{conv}(S)$, $\mathrm{cone}(S)$, $\mathrm{lin}(S)$ denote the convex hull, the convex conical hull and the linear hull of S, respectively. A map $T\colon \mathbf{R}^n \to \mathbf{R}^m$ is called affine if

$$T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y), \qquad x, y \in \mathbf{R}^n,\ \lambda \in \mathbf{R}.$$

In particular, an affine map $f\colon \mathbf{R}^n \to \mathbf{R}$ is called an affine function. It is known that $f\colon \mathbf{R}^n \to \mathbf{R}$ is affine if and only if

$$f(\,\cdot\,) = \langle u,\, \cdot\,\rangle + b$$

for some unique $u \in \mathbf{R}^n$ and $b \in \mathbf{R}$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the classical inner product. An affine function $f(\,\cdot\,) = \langle u,\, \cdot\,\rangle + b$ is called non-trivial (or invertible) if $u \neq o$, the origin or the zero vector. The set of affine functions on $\mathbf{R}^n$ is denoted by $\mathrm{aff}(\mathbf{R}^n)$. In this paper, affine functions always mean the ones in $\mathrm{aff}(\mathbf{R}^n)$ unless other mentioned.

For $o \neq u \in \mathbf{R}^n$ and $b \in \mathbf{R}$, the set $H_{u,b} := \{x \in \mathbf{R}^n \mid \langle u, x\rangle = b\}$ is a hyperplane, and the set $\overline{H}^-_{u,b} := \{x \in \mathbf{R}^n \mid \langle u, x\rangle \leqslant b\}$ (resp. $H^-_{u,b} := \{x \in \mathbf{R}^n \mid \langle u, x\rangle < b\}$) and $\overline{H}^+_{u,b} := \{x \in \mathbf{R}^n \mid \langle u, x\rangle \geqslant b\}$ (resp. $H^+_{u,b} := \{x \in \mathbf{R}^n \mid \langle u, x\rangle > b\}$) are called closed (resp. open) half-spaces with outer and inner normal u respectively. It is trivial that $H_{u,b} = \{f = 0\}$, $\overline{H}^-_{u,b} = \{f \leqslant 0\}$ and $\overline{H}^+_{u,b} = \{f \geqslant 0\}$, etc., where the affine function

$$f(\,\cdot\,) = \langle u,\, \cdot\,\rangle - b.$$

Definition 2.1 *For $L \subset H_{u,b}$, the set $H^-_{u,b} \bigcup L$ (resp. $H^+_{u,b} \bigcup L$) is called a partially-closed half-space with outer (resp. inner) normal u. We adopt $\tilde{H}^-_{u,b}$ and $\tilde{H}^+_{u,b}$, as generic symbols, to denote a partially-closed half-space (without specified L).*

Clearly, when $L = \emptyset$ (resp. $L = H_{u,b}$), we get back the open half-space $H^-_{u,b}$ or $H^+_{u,b}$ (resp. the closed half-space $\overline{H}^-_{u,b}$ or $\overline{H}^+_{u,b}$). However, a partially-closed half-space may not even be convex since L may not be.