COMMUNICATIONS IN MATHEMATICAL RESEARCH
34(1)(2018), 15–22

Growth of Solutions of Some Linear Difference Equations with Meromorphic Coefficients

TU HONG-QIANG AND LIU HUI-FANG*
(*College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, 330022*)

Communicated by Ji You-qing

Abstract: In this paper, we investigate the properties of solutions of some linear difference equations with meromorphic coefficients, and obtain some estimates on growth and value distribution of these meromorphic solutions.

Key words: difference equation, meromorphic function, growth of order, exponent of convergence
2010 MR subject classification: 39A10, 30D35
Document code: A
Article ID: 1674-5647(2018)01-0015-08
DOI: 10.13447/j.1674-5647.2018.01.02

1 Introduction and Main Results

In this paper, we use the standard notations and the fundamental results of Nevanlinna's theory (see [1]–[2]). Let f be a meromorphic function in the whole complex plane, we denote by $\sigma(f)$, $\lambda(f)$ and $\lambda\left(\frac{1}{f}\right)$ the order, the exponent of convergence of zeros and poles of $f(z)$, respectively.

Nevanlinna's theory has been widely applied to the field of complex difference. Many researchers studied the properties of meromorphic solutions of the following linear difference equation by this theory

$$A_n(z)f(z+c_n)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0, \tag{1.1}$$

where $n \in \mathbf{N}$, c_j $(j=1,\cdots,n)$ are nonzero complex numbers which are different from each other, and obtained lots of results concerning the growth and value distribution of meromorphic solutions of (1.1) (see [3]–[9]). Therein Chiang and Feng[4] considered the

Received date: July 29, 2016.
Foundation item: The NSF (11661044, 11201195) of China, and the NSF (20132BAB201008) of Jiangxi Province.
* **Corresponding author**.
E-mail address: 1518048835@qq.com (Tu H Q), liuhuifang73@sina.com (Liu H F).

case when there is only one dominating coefficient among all entire coefficients of (1.1), and obtained the following result:

Theorem A[4] *Let $A_0(z), \cdots, A_n(z)$ be entire functions. If there exists an integer l $(0 \le l \le n)$ such that*

$$\sigma(A_l) > \max_{\substack{0\le j\le n\\ j\ne l}}\{\sigma(A_j)\},$$

then every meromorphic solution $f(\not\equiv 0)$ of (1.1) *satisfies*

$$\sigma(f) \ge \sigma(A_l) + 1.$$

When most coefficients of (1.1) have the same order, Qi *et al.*[9] studied the properties of meromorphic solutions of the following linear difference equation

$$f(z+n) + \sum_{j=0}^{n-1}\{P_j(\mathrm{e}^{A(z)}) + Q_j(\mathrm{e}^{-A(z)})\}f(z+j) = 0, \tag{1.2}$$

and obtained the following results:

Theorem B[9] *Let $P_j(z)$ and $Q_j(z)$ $(j = 0, 1, \cdots, n-1)$ be polynomials, $A(z)$ be a polynomial of degree $k(\ge 1)$. If*

$$\deg(P_0) > \deg(P_j) \quad \text{or} \quad \deg(Q_0) > \deg(Q_j), \qquad j = 1, \cdots, n-1,$$

then each nontrivial meromorphic solution $f(z)$ with finite order of (1.2) *satisfies*

$$\sigma(f) = \lambda(f - a) \ge k + 1,$$

and so f assumes every nonzero complex value $a \in \mathbf{C}$ infinitely often.

Theorem C[9] *Suppose that the assumptions of Theorem B are satisfied. If $f(z)$ is a nontrivial entire solution with finite order of* (1.2) *that satisfies $\lambda(f) \le k$, then $\sigma(f) = k + 1$.*

Comparing Theorem A with Theorem B and Theorem C, we pose the following questions:

Question 1.1 When all coefficients of (1.1) have the form $P_j(\mathrm{e}^{A(z)}) + Q_j(\mathrm{e}^{-A(z)}) + R_j(z)$ $(j = 0, \cdots, n)$, where P_j, Q_j and A are polynomials, R_j are meromorphic functions, and satisfy $\deg(P_l) > \max\limits_{\substack{0\le j\le n\\ j\ne l}}\{\deg(P_j)\}$ or $\deg(Q_l) > \max\limits_{\substack{0\le j\le n\\ j\ne l}}\{\deg(Q_j)\}$, does the conclusion of Theorem B hold?

Question 1.2 Theorem C provided a criterion which guarantee that each entire solution of (1.2) has the smallest order. Then under the assumptions of Question 1.1, what else condition can guarantee that each meromorphic solution of (1.1) has the smallest order?

In this paper, we investigate the above questions and obtain the following results.

Theorem 1.1 *Let $A_j(z) = P_j(\mathrm{e}^{A(z)}) + Q_j(\mathrm{e}^{-A(z)}) + R_j(z)$ $(j = 0, 1, \cdots, n)$, where $A(z)$ are polynomials with degree $k(\ge 1)$, $P_j(z)$ and $Q_j(z)$ $(j = 0, 1, \cdots, n-1)$ are polynomials, $R_j(z)$ are meromorphic functions of $\sigma(R_j) < k$ and $A_j(z) - R_j(z) \not\equiv 0$. If there exists an integer $l \in \{0, 1, \cdots, n\}$ such that*

$$\deg(P_l) > \deg(P_j) \quad \text{or} \quad \deg(Q_l) > \deg(Q_j), \qquad j = 0, 1, \cdots, n, \ j \ne l,$$