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$\mathcal{P}$-congruence-free Epigroups

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Abstract: Let $\mathcal{P}$ denote the equivalence relation on an epigroup in which any of its classes consists precisely of those elements owning the same pseudo-inverse. The purpose of the paper is to characterize epigroups on which any congruence either contains $\mathcal{P}$ or its join with $\mathcal{P}$ is the identity relation on epigroups. As a special subclass of epigroups, completely 0-simple semigroups having the same property are also described.

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1 Introduction and Preliminaries

An epigroup is a semigroup in which some power of any element lies in a subgroup of the given semigroup. The works by Shevrin were devoted to different structural aspects of epigroup theory (see [1]–[3]). For an element x of a given epigroup, let x^ω be the identity of the subgroup G that contains some power of x, and $\overline{x}$ the group inverse of $xx^\omega (= x^\omega x)$ in G, which is called the pseudo-inverse of x. An epigroup can alternatively be regarded as a unary semigroup with the unary operation of taking pseudo-inverse $x \mapsto \overline{x}$, due to Shevrin (see also [1] and [2]). Further new result on epigroups as unary semigroups about this operation, for instance, occurred in [4]. In [5], the equivalence relation $\mathcal{P}$ on an epigroup, in which its arbitrary classes consist precisely of those elements having the same pseudo-inverse, was introduced, and several characterizations of epigroups in which $\mathcal{P}$ is a congruence were also given. In [6], the relation on the congruence lattice of an epigroup that identifies two congruences if they have the same join with $\mathcal{P}$ was considered. In fact, for a better understanding of congruences on some special semigroups, an essential role is played by the

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interplay of an arbitrary congruence with some known equivalence relations, such as Green's relations $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{J}$, which leads to various decompositions of the congruence lattice of a semigroup, helping us to gain insight into the structure of the congruence lattice, even some information on the semigroups. One direction of this idea was reflected in [7], where the authors classed the completely regular semigroups on which any congruence either contains some Green's relation $\mathcal{G}$, or its join with $\mathcal{G}$ is the identity relation on the semigroups.

The lattice of congruences of a semigroup S is denoted by $\mathcal{C}(S)$, which is partially ordered by inclusion. Recall that for an equivalence relation τ on S, S is called to be τ-congruence-free if for any $\rho \in \mathcal{C}(S)$, either $\rho \cap \tau = \Delta$ or $\tau \subseteq \rho$ (see Definition 1.1 in [7]).

In this paper we characterize $\mathcal{P}$-congruence-free epigroups. In Section 2, we deal with $\mathcal{P}$-congruence-free completely 0-simple semigroups and classify them. The main result formulated in Section 3 provides the classification and description of $\mathcal{P}$-congruence-free epigroups. In Section 1, besides required background information concerning epigroups, we present basic information and auxiliary facts about $\mathcal{P}$-congruence-free epigroups. Here we generally follow the notation and terminology of [8] and [9]; in particular, we recall the following concepts.

An element a' in a semigroup S is an inverse of a in S if $aa'a = a$ and $a'aa' = a'$. The set of inverses of a is denoted by $V(a)$, and a is regular if $V(a)$ is not empty. The set of all completely regular elements of a semigroup S (i.e., the group part of S) is denoted by GrS. If S has a zero, Nil$S = \{s \in S : s^n = 0 \text{ for some } n \in \mathbf{Z}^+\}$. For an epigroup S, the equivalence relation $\mathcal{P}$ on S is defined by

$$a\mathcal{P}b \qquad \text{if } \overline{a} = \overline{b} \ (\Leftrightarrow \overline{\overline{a}} = \overline{\overline{b}}\,).$$

Notice that for any $a \in S$, $\overline{\overline{a}} = aa^{\omega}$, then $\overline{a} = \overline{\overline{\overline{a}}}$, so that $a\mathcal{P}\overline{\overline{a}}$ and no $\mathcal{P}$-class contains more than one completely regular element.

The following Lemma is a corollary of Theorem 6.45 in [10].

Lemma 1.1 *Let S be an epigroup.*

(i) *$\mathcal{J} = \mathcal{D}$ in S;*

(ii) *for $a \in S$ and $x \in S^1$, if $a\mathcal{D}xa$, then $a\mathcal{L}xa$;*

(iii) *for $a \in S$ and $y \in S^1$, if $a\mathcal{D}ay$, then $a\mathcal{R}ay$.*

Let ρ be a binary relation on a semigroup S. ρ^{∞} denotes the transitive closure of ρ, and ρ^* denotes the congruence on S generated by ρ. For the latter, explicitly, for $a, b \in S$,

$$a\rho^* b \Leftrightarrow a = b \quad \text{or} \quad a = x_1u_1y_1, x_1v_1y_1 = x_2u_2y_2, \cdots, x_nv_ny_n = b \tag{1.1}$$

for some $x_i, y_i \in S^1$, $u_i, v_i \in S$ such that either $u_i(\rho \cup \Delta)v_i$ or $v_i(\rho \cup \Delta)u_i$, $i = 1, 2, \cdots, n$. For any $a, b \in S$, let $\zeta_{a,b}$ be the congruence on S generated by the singleton $\{(a, b)\}$ (this denotation differs slightly from that of its original notation introduced in [11]). The radical of ρ, in notation $\sqrt{\rho}$, which is due to Shevrin, is a relation on S defined by the following condition:

$$a\sqrt{\rho}\, b \ \Leftrightarrow\ \text{there exist } m, n \in \mathbf{Z}^+ \text{ such that } a^m \rho\, b^n \quad (a, b \in S).$$

The division relation $\mid$ and the relation $\longrightarrow$ on S are defined as follows:

$$a \mid b \ \text{ if } \ b \in J(a) = S^1aS^1, \quad a \longrightarrow b \ \text{ if } \quad a \mid b^n \text{ for some } \ n \in \mathbf{Z}^+ \quad (a, b \in S).$$