Analysis in Theory and Applications
Anal. Theory Appl., Vol. **35**, No. 4 (2019), pp. 392-404

DOI: 10.4208/ata.OA-2018-0006

On the Degree of Approximation of Continuous Functions by Means of Fourier Series in the Hölder Metric

Xhevat Z. Krasniqi[1,*] and Bogdan Szal[1]

[1] *University of Prishtina, Faculty of Education, Department of Mathematics and Informatics, Avenue Mother Theresa, 10000 Prishtina, Kosova*
[2] *University of Zielona Góra, Faculty of Mathematics, Computer Science and Econometrics, 65-516 Zielona Góra, ul. Szafrana 4a, Poland*

Received 8 February 2018; Accepted (in revised version) 21 December 2018

Abstract. In this paper we prove two theorems on the degree of approximation of continuous functions by matrix means related to partial sums of a Fourier series in the Hölder metric. These theorems can be taken as counterparts of those previously obtained by T. Singh [3].

Key Words: Matrix transformation, degree of approximation, Fourier series, Hölder metric.
AMS Subject Classifications: 42A24, 42B05, 42B08

1 Introduction and the aim of the paper

The space of all 2π- periodic continuous functions f on $[0,2\pi]$ with Fourier series

$$f(x)\sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

is denoted by $C_{2\pi}$.

The space H_ω is defined by

$$H_\omega=\{f\in C_{2\pi}:|f(x)-f(y)|\le K\omega(|x-y|)\},$$

while the norm $\|\cdot\|_{\omega^*}$ is defined by

$$\|f\|_{\omega^*}=\|f\|_C+\sup_{x,y}\Delta^{\omega^*}f(x,y),$$

*Corresponding author. *Email addresses:* xhevat.krasniqi@uni-pr.edu (Xh. Z. Krasniqi), B.Szal@wmie.uz.zgora.pl (B. Szal)

where

$$\|f\|_C = \sup_{0\le x\le 2\pi} |f(x)|,$$
$$\Delta^{\omega^*} f(x,y) = \frac{|f(x)-f(y)|}{\omega^*(|x-y|)}, \qquad x\neq y,$$

and $\Delta^0 f(x,y)=0$.

The functions $\omega(t)$ and $\omega^*(t)$ are assumed to be increasing functions of t. If $\omega(|x-y|)\le K_1|x-y|^\alpha$ and $\omega^*(|x-y|)\le K_2|x-y|^\beta$, $0\le\beta<\alpha\le 1$, where K_1,K_2 are positive constants, then the space

$$H_\alpha=\{f\in C_{2\pi}: |f(x)-f(y)|\le K|x-y|^\alpha,\ 0<\alpha\le 1\}$$

is a Banach space [8] and the metric induced by the norm $\|\cdot\|_\alpha$ on H_α is said to be a Hölder metric.

Let $S_n(f;x)$ denote the n-th partial sum of Fourier series of the function f. Let $A:=(a_{n,k})$ $(k,n=0,1,\cdots)$ be a lower triangular infinite matrix of real numbers and let the A-transform of $\{S_n(f;x)\}$ be given by

$$T_{n,A}(f):=T_{n,A}(f;x):=\sum_{k=0}^{n} a_{n,k}S_k(f;x), \quad (n=0,1,\cdots).$$

The following notations will be used later:

$$\phi_x(t):=f(x+t)+f(x-t)-2f(x),$$
$$D_{n,A}(f):=T_{n,A}(f)-f.$$

Seemingly, was Chandra [1] who for the first time extended Prössdorf's [8] results to find the degree of approximation of a continuous function using the Nörlund transform. Later on, Mohapatra and Chandra [2] obtained a number of interesting results on the degree of approximation in the Hölder metric using matrix transforms, which generalize all the previous results based on Cesàro and Nörlund transforms. Their result can be read as follows:

Theorem 1.1. *Let $A:=(a_{n,k})$ $(k,n=0,1,\cdots)$ be a lower triangular infinite matrix such that*

$$a_{n,k}\ge 0,\quad n,k=0,1,2,\cdots,\quad \text{and}\quad \sum_{k=0}^{n}a_{n,k}=1, \tag{1.1a}$$
$$a_{n,k}\le a_{n,k+1},\quad k=0,1,2,\cdots,n-1;\quad n=0,1,2,\cdots. \tag{1.1b}$$

Then for $f\in H_\alpha$, $0\le\beta<\alpha\le 1$

$$\|D_{n,A}(f)\|_\beta=\begin{cases}\mathcal{O}(n^{\beta-\alpha})+\mathcal{O}(a_{n,n}n^{\beta-\alpha+1}), & 0<\alpha<1,\\ \mathcal{O}(n^{\beta-1})+\mathcal{O}(a_{n,n}n^{\beta}(\log n)^{1-\beta}), & \alpha=1.\end{cases}$$

In 1991, Singh [3] generalized the above result under more general assumptions on the function, and obtained a number of interesting corollaries and deductions. Precisely, he proved the following two theorems:

Theorem 1.2. *Let* $A := (a_{n,k})$ $(k,n = 0,1,\cdots)$ *be a lower triangular infinite matrix such that* (1.1a) *and* (1.1b) *hold. Then for* $f \in H_\omega$, $0 \le \beta < \eta \le 1$,

$$\begin{aligned}\|D_{n,A}(f)\|_{\omega^*} =& \mathcal{O}\Bigg\{\sup_{x,y} \frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\\ &\times \left[\left(\omega\left(\frac{1}{n}\right)\right)^{1-\beta/\eta} + a_{n,n}n^{\beta/\eta}\left(\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\right)^{1-\beta/\eta}\right]\Bigg\}.\end{aligned}$$

Theorem 1.3. *Let* $A := (a_{n,k})$ $(k,n = 0,1,\cdots)$ *be a lower triangular infinite matrix such that* (1.1a) *and* (1.1b) *hold. Let* $\omega(t)$ *be the modulus of continuity of* f *such that*

$$\int_u^\pi t^{-2}\omega(t)dt = \mathcal{O}(H(u)), \tag{1.2a}$$

$$\int_0^u H(t)dt = \mathcal{O}(uH(u)), \quad (u \to 0^+), \quad H(u) \ge 0. \tag{1.2b}$$

Then for $f \in H_\omega$, $0 \le \beta < \eta \le 1$,

$$\begin{aligned}\|D_{n,A}(f)\|_{\omega^*} =& \mathcal{O}\Bigg\{\sup_{x,y} \frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\\ &\times \left(a_{n,n}H(a_{n,n})\right)^{1-\beta/\eta}\left(\log(1+na_{n,n})\right)^{\beta/\eta}\Bigg\}.\end{aligned}$$

Later again, Singh [4] introduced H_ω-space in place of H_α-space and obtained several results on the degree of approximation of functions and deduced many previous results based on H_α-spaces. Mittal and Rhoades [6] also obtained the degree of approximation of functions in a normed space and generalized the results of Singh [4] by removing the hypothesis of monotonicity of the rows of the matrix. Very recently T. Singh and P. Mahajan [5] have obtained some further interesting results in this direction. In an attempt to make an advance study in this direction, two new estimates for degree of approximation of continuous functions in Hölder metric is determined, which is the main aim of this paper. These estimates proved in this paper can be taken as counterparts of those obtained in Theorems 1.2 and 1.3 in which we have removed condition (1.1b).

Now and further for an arbitrary sequence $\{b_n\}$ denote

$$\Delta b_n = b_n - b_{n+1} \quad \text{and} \quad \Delta^\nu b_n = \Delta\left(\Delta^{\nu-1}b_n\right), \qquad \nu = 2,3,\cdots.$$

Closing this section we emphasize here that throughout of this paper we write $u = \mathcal{O}_s(v)$ if there exists a finite positive constant $K(s)$, depending on a fixed natural number s, such that $u \le K(s)v$.

2 Auxiliary statements

To prove the main results we need some auxiliary statements.

Lemma 2.1 ([1]). *If* (1.2a) *and* (1.2b) *hold then*

$$\int_0^r t^{-1}\omega(t)dt = \mathcal{O}\left(rH(r)\right), \qquad (r \to +0).$$

Lemma 2.2. *For any lower triangular infinite matrix* $(a_{n,k})$*,* $k,n = 0,1,2,\cdots$*, of nonnegative numbers, it holds uniformly in* $0 < t \le \pi$*, that*

$$\sum_{k=0}^{n} a_{n,k} \sin\left(k+\frac{1}{2}\right)t = \mathcal{O}_s\left(\frac{1}{t}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\right), \tag{2.1}$$

where throughout s *is fixed and* $s \in \{1,2,\cdots\}$*.*

Proof. For arbitrary $\lambda_n \ge 0$ and for $n \ge m \ge 0$ we have

$$\begin{aligned}
\Lambda_{m,n} :=& \sum_{k=m}^{n} \lambda_k \sin\left(k+\frac{1}{2}\right)t \sin\frac{t}{2} \\
=&\frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \Delta\lambda_k \cos(k+1)t - \lambda_n \cos(n+1)t\right] \\
=&\frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \Delta\lambda_k \cos\left(k+\frac{1}{2}\right)t\cos\frac{t}{2}\right. \\
&\left. + \sum_{k=m}^{n-1} \Delta\lambda_k \sin\left(k+\frac{1}{2}\right)t\sin\frac{t}{2} - \lambda_n\cos(n+1)t\right] \\
=&\frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \Delta\lambda_k \cos\left(k+\frac{1}{2}\right)t\cos\frac{t}{2} - \lambda_n\cos(n+1)t\right] \\
&+\frac{1}{2^2}\left[\Delta\lambda_m \cos mt - \sum_{k=m}^{n-2} \Delta^2\lambda_k \cos(k+1)t - \Delta\lambda_{n-1}\cos nt\right].
\end{aligned}$$

Repeating this transformation, in the same way $s-$times, we easy obtain

$$\begin{aligned}
\Lambda_{m,n} :=&\frac{1}{2}\left[\lambda_m \cos mt - \sum_{k=m}^{n-1} \Delta\lambda_k \cos\left(k+\frac{1}{2}\right)t\cos\frac{t}{2} - \lambda_n\cos(n+1)t\right] \\
&+\frac{1}{2^2}\left[\Delta\lambda_m \cos mt - \sum_{k=m}^{n-2} \Delta^2\lambda_k \cos(k+1)t - \Delta\lambda_{n-1}\cos nt\right] + \cdots \\
&+\frac{1}{2^s}\left[\Delta^{s-1}\lambda_m \cos mt - \sum_{k=m}^{n-s} \Delta^s\lambda_k \cos(k+1)t - \Delta^{s-1}\lambda_{n-s+1}\cos(n-s)t\right],
\end{aligned}$$

where $s \in \{1,2,\cdots\}$. Thus,

$$|\Lambda_{m,n}| \leq \frac{1}{2}\left(\lambda_m + \sum_{k=m}^{n-1}|\Delta\lambda_k| + \lambda_n\right) + \frac{1}{2^2}\left(|\Delta\lambda_m| + \sum_{k=m}^{n-2}|\Delta^2\lambda_k| + |\Delta\lambda_{n-1}|\right) + \cdots + \frac{1}{2^s}\left(|\Delta^{s-1}\lambda_m| + \sum_{k=m}^{n-s}|\Delta^s\lambda_k| + |\Delta^{s-1}\lambda_{n-s+1}|\right), \tag{2.2}$$

where $s \in \{1,2,\cdots\}$.

But since (a_{nk}) is a lower triangular matrix, then

$$a_{n,m} \leq \sum_{k=m}^{n}|\Delta a_{n,k}|, \quad |\Delta a_{n,m}| \leq \sum_{k=m}^{n}|\Delta^2 a_{n,k}|, \cdots, |\Delta^{s-1}a_{n,m}| \leq \sum_{k=m}^{n}|\Delta^s a_{n,k}|, \tag{2.3}$$

hold for $n-s+1 \geq m \geq 0$. Now by (2.2) and (2.3) we have

$$\begin{aligned}&\left|\sum_{k=0}^{n} a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right| \\ =&\mathcal{O}_s\left(\frac{1}{t}\left(\left(a_{n,0} + \sum_{k=0}^{n-1}|\Delta a_{n,k}| + a_{n,n}\right) + \left(|\Delta a_{n,0}| + \sum_{k=0}^{n-2}|\Delta^2 a_{n,k}| + |\Delta a_{n,n-1}|\right)\right.\right. \\ &\left.\left. + \cdots + \left(|\Delta^{s-1}a_{n,0}| + \sum_{k=0}^{n-s}|\Delta^s a_{n,k}| + |\Delta^{s-1}a_{n,n-s+1}|\right)\right)\right) \\ =&\mathcal{O}_s\left(\frac{1}{t}\sum_{k=0}^{n}\left(|\Delta a_{n,k}| + |\Delta^2 a_{n,k}| + \cdots + |\Delta^s a_{n,k}|\right)\right),\end{aligned}$$

which completes (2.1), and with this the proof of the lemma. □

3 Main results

At the beginning we shall prove a generalization of Theorem 1.2.

Theorem 3.1. *Let $A := (a_{n,k})$ $(k,n = 0,1,\cdots)$ be a lower triangular infinite matrix such that conditions* (1.1a) *are satisfied. Then for $f \in H_\omega$, $0 \leq \beta < \eta \leq 1$,*

$$\|D_{n,A}(f)\|_{\omega^*} = \mathcal{O}_s\left(\sup_{x,y}\frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\left[\left(\omega\left(\frac{1}{n}\right)\right)^{1-\beta/\eta} + n^{\beta/\eta}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\left(\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\right)^{1-\beta/\eta}\right]\right). \tag{3.1}$$

Proof. We shall follow the idea of T. Singh [3]. A straightforward calculation gives

$$|\phi_x(t)-\phi_y(t)|\le K\omega(|t|), \tag{3.2a}$$
$$|\phi_x(t)-\phi_y(t)|\le K\omega(|x-y|). \tag{3.2b}$$

Then we easy obtain

$$\begin{aligned}D_{n,A}(x):=&T_{n,A}(f;x)-f(x)\\ =&\frac{1}{\pi}\int_0^{\pi}\phi_x(t)\left(2\sin\frac{t}{2}\right)^{-1}\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)tdt\end{aligned}$$

and hence

$$\begin{aligned}|D_{n,A}(x,y)|:=&|D_{n,A}(x)-D_{n,A}(y)|\\ =&\mathcal{O}(1)\int_0^{\pi}\frac{|\phi_x(t)-\phi_y(t)|}{t}\left|\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|dt\\ =&\mathcal{O}(1)\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right)\\ =:&J_1+J_2.\end{aligned}$$

Now using (1.1a) and (3.2) we obtain

$$\begin{aligned}J_1=&\mathcal{O}(1)\int_0^{\pi/n}\frac{|\phi_x(t)-\phi_y(t)|}{t}\sum_{k=0}^{n}a_{n,k}\left(k+\frac{1}{2}\right)tdt\\ =&\mathcal{O}(n)\int_0^{\pi/n}\omega(|t|)dt=\mathcal{O}(\omega(\pi/n)),\end{aligned} \tag{3.3}$$

while Lemma 2.2 yields

$$\begin{aligned}J_2=&\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\right)\int_{\pi/n}^{\pi}\frac{\omega(t)}{t^2}dt\\ =&\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\right)\int_{1/\pi}^{n/\pi}\omega(1/t)dt\\ =&\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\right)\sum_{k=1}^{n-1}\int_{k/\pi}^{(k+1)/\pi}\omega(1/t)dt\\ =&\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\sum_{k=0}^{n}\omega(1/k)\right).\end{aligned} \tag{3.4}$$

Using (1.1a), (3.2b), and Lemma 2.2 we also obtain

$$J_1=\mathcal{O}_s\left(\int_0^{\pi/n}\frac{\omega(|x-y|)}{t}\sum_{k=0}^{n}a_{n,k}\left(k+\frac{1}{2}\right)tdt\right)=\mathcal{O}_s(\omega|x-y|),$$

and

$$\begin{aligned} J_2 =& \mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\right)\int_{\pi/n}^{\pi}\frac{\omega(|x-y|)}{t^2}dt \\ =& \mathcal{O}_s\left(n\omega(|x-y|)\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\right). \end{aligned}$$

Next we can write

$$J_1 = J_1^{1-\beta/\eta}J_1^{\beta/\eta} = \mathcal{O}_s\left((\omega(|x-y|))^{\beta/\eta}(\omega(1/n))^{1-\beta/\eta}\right)$$

and in a same way

$$\begin{aligned} J_2 =& J_2^{1-\beta/\eta}J_2^{\beta/\eta} \\ =& \mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|(n\omega(|x-y|))^{\beta/\eta}\left(\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\right)^{1-\beta/\eta}\right). \end{aligned}$$

Thus

$$\begin{aligned} &\sup_{x,y}|\Delta^{\omega^*}D_{n,A}(x,y)| \\ =&\sup_{x,y}\frac{|D_{n,A}(x)-D_{n,A}(y)|}{\omega^*(|x-y|)} = \mathcal{O}_s\Bigg\{\sup_{x,y}\frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)} \\ &\times\left[\left(\omega\left(\frac{1}{n}\right)\right)^{1-\beta/\eta} + n^{\beta/\eta}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\left(\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\right)^{1-\beta/\eta}\right]\Bigg\}. \end{aligned}$$

Moreover, from (3.3) and (3.4) it is clear that

$$\begin{aligned} &\|D_{n,A}(x)\|_C = \max_{0\le x\le 2\pi}|T_{n,A}(f;x)-f(x)| \\ =&\mathcal{O}_s\left(\omega\left(\frac{1}{n}\right)+\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\sum_{k=0}^{n}\omega\left(\frac{1}{k}\right)\right) \\ =&\mathcal{O}_s\left((\omega(1))^{\beta/\eta}\left[\left(\omega\left(\frac{1}{n}\right)\right)^{1-\beta/\eta}+n^{\beta/\eta}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|\left(\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\right)^{1-\beta/\eta}\right]\right). \end{aligned}$$

Combining last two estimates we obtain (3.1). The proof is completed. □

Theorem 3.2. *Let $A := (a_{n,k})$ $(k,n = 0,1,\cdots)$ be a lower triangular infinite matrix such that* (1.1a) *holds. Let $\omega(t)$ be the modulus of continuity of f such that* (1.2a) *and* (1.2b) *are satisfied. Then for $f \in H_\omega$, $0 \le \beta < \eta \le 1$,*

$$\begin{aligned}&\|D_{n,A}(f)\|_{\omega^*}\\ =&\mathcal{O}_s\left(\sup_{x,y}\frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\times\left[\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|H\left(\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right)\right)^{1-\beta/\eta}\right.\right.\\ &\left.\left.\times\left(\ln\left[1+(n+1)\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right]\right)^{\beta/\eta}\right]\right).\end{aligned}$$

Proof. Since

$$\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}| \le \frac{2+2^2+\cdots+2^s}{2^s-1}\sum_{k=0}^{n}a_{nk} = 2 < \pi,$$

then we can decompose the integral in this form

$$\begin{aligned}&|D_{n,A}(x,y)| = |D_{n,A}(x) - D_{n,A}(y)|\\ =&\mathcal{O}(1)\left(\int_0^{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|} + \int_{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}^{\pi}\right)\\ =&:H_1 + H_2.\end{aligned}$$

Applying Lemma 2.1, we have

$$\begin{aligned}H_1 =&\mathcal{O}(1)\int_0^{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}\frac{\omega(t)}{t}\left|\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|dt\\ =&\mathcal{O}_s(1)\int_0^{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}t^{-1}\omega(t)dt\\ =&\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|H\left(\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right)\right).\end{aligned}\tag{3.5}$$

In order to estimate the second integral we apply Lemma 2.2:

$$\begin{aligned}H_2 =&\mathcal{O}(1)\int_{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}^{\pi}\frac{\omega(t)}{t}\left|\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|dt\\ =&\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right)\int_{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}^{\pi}t^{-2}\omega(t)dt\\ =&\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|H\left(\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right)\right).\end{aligned}\tag{3.6}$$

Since $a_{nk}=0$ for $k>n$, we deduce that

$$a_{n\ell}\leq\sum_{k=\ell}^{n}|\Delta a_{nk}|\leq\sum_{k=0}^{n}\left(|\Delta a_{nk}|+|\Delta^2 a_{nk}|+\cdots+|\Delta^s a_{nk}|\right)$$

for $\ell=0,1,2,\cdots,n$, which implies

$$1=\sum_{\ell=0}^{n}a_{n\ell}\leq(n+1)\sum_{k=0}^{n}\left(|\Delta a_{nk}|+|\Delta^2 a_{nk}|+\cdots+|\Delta^s a_{nk}|\right),$$

i.e.,

$$\sum_{k=0}^{n}\left(|\Delta a_{nk}|+|\Delta^2 a_{nk}|+\cdots+|\Delta^s a_{nk}|\right)\geq\frac{1}{n+1}.$$

Hence using (3.2b) and last inequality we have

$$\begin{aligned}H_1=&\mathcal{O}_s(\omega(|x-y|))\int_0^{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}\frac{1}{t}\left|\sum_{k=0}^{n}a_{n,k}\sin\left(k+\frac{1}{2}\right)t\right|dt\\=&\mathcal{O}_s(\omega(|x-y|))\left(\int_0^{\frac{1}{2^s-1}\frac{1}{n+1}}(n+1)\,dt+\int_{\frac{1}{2^s-1}\frac{1}{n+1}}^{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}t^{-1}dt\right)\\=&\mathcal{O}_s\left(\omega(|x-y|)\left(1+\ln\left[(n+1)\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right]\right)\right)\\=&\mathcal{O}_s\left(\omega(|x-y|)\left(\ln\left[1+(n+1)\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right]\right)\right)\end{aligned}$$

and

$$\begin{aligned}H_2=&\mathcal{O}_s\left(\omega(|x-y|)\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right)\int_{\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|}^{\pi}t^{-2}dt\\=&\mathcal{O}_s\left(\omega(|x-y|)\right).\end{aligned}$$

Therefore we can write

$$\begin{aligned}H_1=&H_1^{1-\beta/\eta}H_1^{\beta/\eta}=\left[\mathcal{O}_s\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|H\left(\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right)\right)\right]^{1-\beta/\eta}\\&\times\left[\mathcal{O}_s\left(\omega(|x-y|)\left(\ln\left[1+(n+1)\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right]\right)\right)\right]^{\beta/\eta}\\=&\mathcal{O}_s\Bigg\{(\omega(|x-y|))^{\beta/\eta}\left(\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|H\left(\frac{1}{2^s-1}\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right)\right)^{1-\beta/\eta}\\&\times\left(\ln\left[1+(n+1)\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}|\right]\right)^{\beta/\eta}\Bigg\},\end{aligned}$$

and similarly we have obtained

$$H_2 = \mathcal{O}_s \left\{ (\omega(|x-y|))^{\beta/\eta} \left(\sum_{k=0}^{n} \sum_{j=1}^{s} |\Delta^j a_{nk}| H \left(\frac{1}{2^s-1} \sum_{k=0}^{n} \sum_{j=1}^{s} |\Delta^j a_{nk}| \right) \right)^{1-\beta/\eta} \right\}.$$

Thus we have

$$\begin{aligned}
&\sup_{x,y} |\Delta^{\omega^*} D_{n,A}(x,y)| \\
=&\mathcal{O}_s \left\{ \sup_{x,y} \frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)} \left(\sum_{k=0}^{n} \sum_{j=1}^{s} |\Delta^j a_{nk}| H \left(\frac{1}{2^s-1} \sum_{k=0}^{n} \sum_{j=1}^{s} |\Delta^j a_{nk}| \right) \right)^{1-\beta/\eta} \right. \\
&\qquad \left. \times \left(\ln \left[1 + (n+1) \sum_{k=0}^{n} \sum_{j=1}^{s} |\Delta^j a_{nk}| \right] \right)^{\beta/\eta} \right\}. \qquad (3.7)
\end{aligned}$$

Moreover, (3.5) and (3.6) imply that

$$\|D_{n,A}(x)\|_C = \mathcal{O}_s \left(\sum_{k=0}^{n} \sum_{j=1}^{s} |\Delta^j a_{nk}| H \left(\frac{1}{2^s-1} \sum_{k=0}^{n} \sum_{j=1}^{s} |\Delta^j a_{nk}| \right) \right). \qquad (3.8)$$

Finally, (3.7) and (3.8) completely proves the theorem. □

4 Corollaries

In this section we are going to show some particular results following from the main results. In [7], L. Leindler has introduced a new class of sequences the so-called $\gamma RBVS$ class. That definition can be stated as follows:

For a fixed n, let $\gamma_n := \{\gamma_{n,k}\}$, $(k = 0, 1, \cdots)$ be a nonnegative sequence. If a null-sequence $\alpha_n := \{a_{n,k}\}$, $(k = 0, 1, \cdots)$ of real numbers has the property

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq K(\alpha_n)\gamma_{n,m}$$

for every positive integer m, then we call the sequence $\alpha_n := \{a_{n,k}\}$ a $\gamma RBVS$, briefly denoted by $\alpha_n \in \gamma RBVS$.

Similarly, in [9] was introduced a new kind of sequences $\gamma HBVS$ as follows:

For a fixed n, let $\gamma_n := \{\gamma_{n,k}\}$, $(k = 0, 1, \cdots)$ be a nonnegative sequence. If a null-sequence $\alpha_n := \{a_{n,k}\}$, $(k = 0, 1, \cdots)$ of real numbers has the property

$$\sum_{k=0}^{m-1} |a_{n,k} - a_{n,k+1}| \leq K(\alpha_n)\gamma_{n,m}$$

for every positive integer m, then we call the sequence $\alpha_n := \{a_{n,k}\}$ a $\gamma HBVS$, briefly denoted by $\alpha_n \in \gamma HBVS$.

It is obvious that if $\gamma_n = \alpha_n$, then $\gamma RBVS \equiv RBVS$ and $\gamma HBVS \equiv HBVS$. For $s = 1$ the same reasoning as in [9] implies that, if $(a_{n,k}) \in HBVS$ then

$$\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}| \leq (K(\alpha_n)+1)a_{n,n},$$

and if $(a_{n,k}) \in RBVS$ then

$$\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{nk}| \leq K(\alpha_n)a_{n,0}.$$

Thus, the above inequalities imply:

Corollary 4.1. *Under the same conditions as in Theorem 3.1 and $(a_{n,k}) \in HBVS$ it holds*

$$\begin{aligned}&\|D_{n,A}(f)\|_{\omega^*}\\ =&\mathcal{O}\left\{\sup_{x,y}\frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\times\left[\left(\omega\left(\frac{1}{n}\right)\right)^{1-\beta/\eta}\right.\right.\\ &\left.\left.+n^{\beta/\eta}a_{n,n}\left(\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\right)^{1-\beta/\eta}\right]\right\}.\end{aligned}$$

Corollary 4.2. *Under the same conditions as in Theorem 3.1 and $(a_{n,k}) \in RBVS$ it holds*

$$\begin{aligned}&\|D_{n,A}(f)\|_{\omega^*}\\ =&\mathcal{O}\left\{\sup_{x,y}\frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\times\left[\left(\omega\left(\frac{1}{n}\right)\right)^{1-\beta/\eta}\right.\right.\\ &\left.\left.+n^{\beta/\eta}a_{n,0}\left(\sum_{k=1}^{n}\omega\left(\frac{1}{k}\right)\right)^{1-\beta/\eta}\right]\right\}.\end{aligned}$$

Corollary 4.3. *Under the same conditions as in Theorem 3.2 and $(a_{n,k}) \in HBVS$ it holds*

$$\begin{aligned}&\|D_{n,A}(f)\|_{\omega^*}\\ =&\mathcal{O}\left\{\sup_{x,y}\frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\big(a_{n,n}H(a_{n,n})\big)^{1-\beta/\eta}\right.\\ &\left.\times\Big(\ln\big[1+(n+1)a_{n,n}\big]\Big)^{\beta/\eta}\right\}.\end{aligned}$$

Corollary 4.4. *Under the same conditions as in Theorem 3.2 and* $(a_{n,k}) \in RBVS$ *it holds*

$$\begin{aligned}&\|D_{n,A}(f)\|_{\omega^*}\\ =&\mathcal{O}\left\{\sup_{x,y}\frac{(\omega(|x-y|))^{\beta/\eta}}{\omega^*(|x-y|)}\big(a_{n,0}H(a_{n,0})\big)^{1-\beta/\eta}\right.\\ &\left.\times\Big(\ln\big[1+(n+1)a_{n,0}\big]\Big)^{\beta/\eta}\right\}.\end{aligned}$$

5 Conclusions

As is said, with reason, in many papers approximation by trigonometric polynomials is in the unit of approximation theory. This is owing to periodicity and continuity of the functions. Such approximation has been realized, in different metrics and particularly in Hölder's metric, for continuous and periodic functions using transformation of partial sums of their Fourier series by a lower triangular infinite matrix of real numbers [2-6] and [10-13]. In many interesting results we have encounter several different type of conditions on the lower triangular infinite matrix (belonging several classes of the real sequences).

In this paper we have presented some new results pertaining to approximation of continuous functions by above mentioned transformation. Moreover, some corollaries of the main result are deduced. These results can be taken not only as counterparts of some earlier results proved by others but also as their significant generalizations. Among others, we have used in the degree of approximation the quantity

$$\sum_{k=0}^{n}\sum_{j=1}^{s}|\Delta^j a_{n,k}|, \quad s\in\{1,2,\cdots\}, \tag{5.1}$$

which clearly is a generalization of the quantity

$$\sum_{k=0}^{n}|\Delta a_{n,k}|$$

as well as $a_{n,n}$ and $a_{n,0}$. Finally, we can freely say that many papers published so far can be also generalized using the quantity (5.1) and having further new results on this treated topic.

References

[1] P. Chandra, On the degree of approximation of a class of functions by means of Fourier series, Acta Math. Hungar., 52 (1988), 199–205.

[2] R. N. Mohapatra and P. Chandra, Degree of approximation of functions in the Hölder metric, Acta Math. Hungar., 41(1-2) (1983), 67–76.

[3] T. Singh, Approximation to continuous in the Hölder metric, Mat. Vesnik, 43 (1991), 111–118.
[4] T. Singh, Degree of approximation to functions in a normed space, Publ. Math. Debrecen, 40(3-4) (1992), 261—271.
[5] T. Singh and P. Mahajan, Error bound of periodic signals in the Hölder metric, Int. J. Math. Math. Sci., (2008), Article ID 495075, 9 pages.
[6] M. L. Mittal and B. E. Rhoades, Degree of approximation to functions in a normed space, J. Comput. Anal. Appl., 2(1) (2000), 1–10.
[7] L. Leindler, Embedding results regarding strong approximation of sine series, Acta Sci. Math. (Szeged), 71 (2005), 93–103.
[8] S. Prössdorf, Zur Konvergenz der Fourierreihen Hölderstiger Funktionen, Math. Nachr, 69 (1975), 7–14.
[9] B. Wei and D. Yu, On the degree of approximation of continuous functions by means of Fourier series, Math. Commun., 17 (2012), 211–219.
[10] K. Khatri and V. N. Mishra, Degree of approximation by the $(T.E^1)$ means of conjugate series of Fourier Series in the Hölder metric, Iran J. Sci. Technol. Trans. A Sci., (2018). `https://doi.org/10.1007/s40995-017-0272-3.`
[11] V. N. Mishra and K. Khatri, Degree of approximation of functions $\tilde{f} \in H_\omega$ class by the $(N_p \cdot E^1)$ means in the Hölder metric, Int. J. Math. Math. Sci., (2014), Article ID 837–408.
[12] V. N. Mishra and L. N. Mishra, Trigonometric approximation of signals (functions) in L_p-norm, Int. J. Contemp. Math. Sci., 7(17-20) (2012), 909–918.
[13] V. N. Mishra, Some Problems on Approximations of Functions in Banach Spaces, Ph.D. Thesis (2007), Indian Institute of Technology, Roorke 247 667, Uttarakhand, India.