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A Note on Weak Type $(1,1)$ Estimate for the Higher Order Commutators of Christ-Journé Type

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Abstract. In this paper, a weak type $(1,1)$ estimate is established for the higher order commutator introduced by Christ and Journé which is defined by

$$T[a_1,\cdots,a_l]f(x)=\text{p.v.}\int_{\mathbb{R}^d}K(x-y)\Big(\prod_{i=1}^{l}m_{x,y}a_i\Big)\cdot f(y)dy,$$

where K is the standard Calderón-Zygmund convolution kernel on $\mathbb{R}^d(d\geq 2)$ and $m_{x,y}a_i=\int_0^1 a_i(sx+(1-s)y)ds$.

Key Words: Weak type $(1,1)$, higher order, commutator.

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1 Introduction

Suppose that K is the standard Calderón-Zygmund convolution kernel on $\mathbb{R}^d\setminus\{0\}$, $(d\geq 2)$, which means that K satisfies the following conditions:

$$|K(x)|\leq C|x|^{-d},\quad \int_{R<|x|<2R}K(x)dx=0\quad \text{holds for all } R>0, \tag{1.1a}$$

$$|K(x-y)-K(x)|\leq C|y|^{\delta}|x|^{-d-\delta}\quad \text{for some } 0<\delta\leq 1 \text{ if } |x|>2|y|. \tag{1.1b}$$

In 1987, Christ and Journé [5] introduced a higher dimensional commutator associated with K and $a_i\in L^\infty(\mathbb{R}^d)$ $(i=1,\cdots,l)$ by

$$T[a_1,\cdots,a_l]f(x)=\text{p.v.}\int_{\mathbb{R}^d}K(x-y)\Big(\prod_{i=1}^{l}m_{x,y}a_i\Big)\cdot f(y)dy,\quad f\in\mathcal{S}(\mathbb{R}^d),$$

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where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class and

$$m_{x,y}a_i = \int_0^1 a_i((1-t)x+ty)dt = \int_0^1 a_i(tx+(1-t)y)dt.$$

Note that $T[a_1,\cdots,a_l]f(x)$ can be seen as a higher dimensional generalization of the following commutator

$$\text{p.v.}\int_{\mathbb{R}} \prod_{i=1}^{l} \Big(\frac{A_i(x)-A_i(y)}{x-y}\Big)\frac{f(y)}{x-y}dy,$$

which is the famous Calderón commutator discussed in [3] and is related to the study of the Cauchy integral, boundary value problem of elliptic equation on non-smooth domain (see e.g., [4, 10, 15]).

Observe that the kernel $K(x-y)$ is smooth but $m_{x,y}a_i$ has no smoothness about variable x and y if $a_i \in L^\infty(\mathbb{R}^d)$. Therefore the standard Calderón-Zygmund theory cannot be applied directly. Christ and Journé [5] proved that $T[a_1,\cdots,a_l]$ is bounded on $L^p(\mathbb{R}^d)$ $(1<p<\infty)$ when $a_i \in L^\infty(\mathbb{R}^d)$ $(i=1,\cdots,l)$. In 1995, Hofmann [14] gave the weighted $L^p(\mathbb{R}^d)$ $(1<p<\infty)$ boundedness of $T[a_1,\cdots,a_l]$, when the kernel $K(x)=\Omega(x/|x|)|x|^{-d}$. Recently, there are renew interests on this singular integral of Christ-Journé type since it has some direct applications in the mixing flows problem (see e.g., [2, 13]). In 2015, A. Seeger, C. Smart and B. Street [19] further studied the commutator of Christ-Journé type and established some multilinear estimates. Later, the second author of the present paper established all multilinear estimates of the higher Calderón commutator (see [16]). For the endpoint case $p=1$, the weak type (1,1) estimate seems to be difficulty and the previous result is only known for the first order commutator. In 2012, Grafakos and Honzík [12] proved that the commutator $T[a]$ is of weak type (1,1) for $d=2$. Later, Seeger [18] showed that $T[a]$ is also of weak type (1,1) for all $d \geq 2$. In [6], the authors established weighted L^p boundedness of $T[a]$ for A_p weight with $d \geq 2$ and weighted weak type (1,1) boundedness for power weight $|x|^\alpha(-2<\alpha<0)$ with $d=2$ (later we extended this result to general A_1 weight for all $d \geq 2$ in [8]). However, the weak type (1,1) estimate for the higher order commutator seems to be unexplored and may be very difficult since the kernel involves with more than two rough factors $\prod_{i=1}^l m_{x,y}a_i$ under the condition that all $a_i \in L^\infty(\mathbb{R}^d)(i=1,\cdots,l)$. In this paper, we try to give a weak type (1,1) estimate for $T[a_1,\cdots,a_l]$ with some restricted condition of a_i. Our main result is as follows.

Theorem 1.1. *Suppose K satisfies* (1.1a) *and* (1.1b) *for $d \geq 2$. Let $a_1 \in L^\infty(\mathbb{R}^d)$. Assume $a_i, \widehat{a_i} \in L^1(\mathbb{R}^d)$, $i=2,\cdots,l$. Then there exists a constant $C>0$ such that*

$$m(\{x\in\mathbb{R}^d : |T[a_1,\cdots,a_l]f(x)|>\lambda\}) \leq C\lambda^{-1}\|a_1\|_\infty \Big(\prod_{i=2}^{l}\|\widehat{a_i}\|_1\Big)\|f\|_1$$

for all $\lambda>0$ and $f \in L^1(\mathbb{R}^d)$.