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Some Results on the Growth of Polynomials

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Abstract. In this paper, we have studied the Lacunary type of polynomials and proved a result which generalizes as well as refines some well-known polynomial inequalities regarding the growth of polynomials not vanishing inside a circle. Further the paper corrects the proofs of some already known results.

Key Words: Lacunary polynomial, Growth, Bernstien-inequality.

AMS Subject Classifications: 30A10, 30C15, 26D10

1 Introduction and statement of results

For an entire function $f(z)$, let $M(f,r)=\max_{|z|=r}|f(z)|$ and $m(f,r)=\min_{|z|=r}|f(z)|$. If $P(z)$ is a polynomial of degree n and $P'(z)$ denote its derivative, then according to a famous result of Bernstien [3],

$$M(P',1)\leq nM(P,1). \tag{1.1}$$

Inequality (1.1) is best possible and equality holds for the polynomial having all its zeros at the origin.

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then

$$M(P',1)\leq \frac{n}{2}M(P,1). \tag{1.2}$$

Inequality (1.2) was conjectured by Erdös and later verified by Lax [8] and is best possible for the polynomials having all its zeros on $|z|=1$.

Aziz and Dawood [1] under the same hypothesis refined the inequality (1.2) and proved

$$M(P',1)\leq \frac{n}{2}\Big\{M(P,1)-m(P,1)\Big\}. \tag{1.3}$$

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This result is sharp and equality holds for the polynomial $P(z)=(z+1)^n$.

As an extension of (1.2), Malik [9] proved that if $P(z)$ does not vanish in $|z|<k$, $k\geq 1$, then

$$M(P',1)\leq \frac{n}{1+k}M(P,1), \tag{1.4}$$

whereas Govil [7] under the same hypothesis proved that

$$M(P',1)\leq \frac{n}{1+k}\Big\{M(P,1)-m(P,k)\Big\}. \tag{1.5}$$

In the literature there exist several generalizations and extensions of the above inequalities, for example see [2, 5, 6, 12–15]. As a generalization of inequality (1.4), Dewan and Bidkham [5] proved that if $P(z)$ has no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq \rho\leq k$,

$$M(P',\rho)\leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}M(P,r). \tag{1.6}$$

Recently Dewan and Mir [6] generalized inequality (1.6) and proved that if $P(z)=\sum_{j=0}^n a_jz^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq \rho\leq k$,

$$M(P',\rho)\leq n\frac{(\rho+k)^{n-1}}{(k+r)^n}\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}M(P,r). \tag{1.7}$$

Further, Aziz and Zargar [2] obtained a generalization of (1.7) by proving the following result.

Theorem 1.1. *If $P(z)=\sum_{j=0}^n a_jz^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $0\leq r\leq \rho\leq k$,*

$$\begin{aligned}M(P',\rho)\leq \frac{n}{\rho+k}\Bigg[&\left(\frac{\rho+k}{k+r}\right)^n\left\{1-\frac{k(k-\rho)(n|a_0|-k|a_1|)n}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\left(\frac{\rho-r}{k+\rho}\right)\left(\frac{k+r}{k+\rho}\right)^{n-1}\right\}M(P,r)\\ &-\left\{\frac{(n|a_0|\rho+k^2|a_1|)(r+k)}{n|a_0|(k^2+\rho^2)+2k^2\rho|a_1|}\left(\left(\left(\frac{\rho+k}{r+k}\right)^n-1\right)-n(\rho-r)\right)\right\}m(P,k)\Bigg].\end{aligned} \tag{1.8}$$

The result is best possible and equality holds in (1.8) for $P(z)=(z+k)^n$.

Recently Zireh [12] obtained the following result which is a refinement of (1.8) and proved.