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Maximal Inequalities for the Best Approximation Operator and Simonenko Indices

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Abstract. In an abstract set up, we get strong type inequalities in L^{p+1} by assuming weak or extra-weak inequalities in Orlicz spaces. For some classes of functions, the number p is related to Simonenko indices. We apply the results to get strong inequalities for maximal functions associated to best Φ-approximation operators in an Orlicz space L^Φ.

Key Words: Simonenko indices, maximal inequalities, best approximation.

AMS Subject Classifications: 41A10, 41A50, 41A45

1 Introduction

In this paper we denote by $\mathfrak{I}$ the set of all non decreasing functions φ defined for all real number $x>0$, such that $\varphi(x)>0$ for all $x>0$, $\varphi(0+)=0$ and $\lim\limits_{x\to\infty}\varphi(x)=\infty$.

We say that a non decreasing function $\varphi:\mathbb{R}_0^+\to\mathbb{R}_0^+$ satisfies the Δ_2 condition, symbolically $\varphi\in\Delta_2$, if there exists a constant $\Lambda_\varphi>0$ such that $\varphi(2x)\le\Lambda_\varphi\varphi(x)$ for all $x\ge 0$.

Now, given $\varphi\in\mathfrak{I}$, we consider $\Phi(x)=\int_0^x\varphi(t)dt$. Observe that $\Phi:[0,\infty)\to[0,\infty)$ is a convex function such that $\Phi(x)=0$ if and only if $x=0$. In the literature, a function Φ satisfying the previous conditions is known as a Young function. In addition, as $\varphi\in\mathfrak{I}$ we have that Φ is increasing, $\frac{\Phi(x)}{x}\to 0$ as $x\to 0$ and $\frac{\Phi(x)}{x}\to\infty$ as $x\to\infty$. Thus, according to [6], a function Φ with this property is called an N-function.

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If $\varphi \in \mathcal{I}$ is a right-continuous function that satisfies the Δ_2 condition, then

$$\frac{1}{2}(\varphi(a)+\varphi(b)) \leq \varphi(a+b) \leq \Lambda_\varphi(\varphi(a)+\varphi(b))$$

for every $a,b \geq 0$.

Also note that the Δ_2 condition on Φ implies

$$\frac{x}{2\Lambda_\varphi}\varphi(x) \leq \Phi(x) \leq x\varphi(x)$$

for every $x \geq 0$.

If $\varphi \in \mathcal{I}$, we define $L^\varphi(\mathbb{R}^n)$ as the class of all Lebesgue measurable functions f defined on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n}\varphi(t|f|)dx < \infty$ for some $t > 0$ and where dx denotes the Lebesgue measure on $\mathbb{R}^n$. For a convex function Φ, $L^\Phi(\mathbb{R}^n)$ is the classic Orlicz space (see [10]). And, if $\Phi \in \Delta_2$ then $L^\Phi(\mathbb{R}^n)$ is the space of all measurable functions f defined on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n}\Phi(|f|)dx < \infty$.

A non decreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies the ∇_2 condition, denoted $\varphi \in \nabla_2$, if there exists a constant $\lambda_\varphi > 2$ such that $\varphi(2x) \geq \lambda_\varphi \varphi(x)$ for all $x \geq 0$.

We claim that a non decreasing function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfies the Δ' condition, symbolically $\varphi \in \Delta'$, if there exists a constant $K > 0$ such that $\varphi(xy) \leq K\varphi(x)\varphi(y)$ for all $x,y \geq x_0 \geq 0$. If $x_0 = 0$ then φ satisfies the Δ' condition globally (denoted $\varphi \in \Delta'$ globally).

With the aim of comparing functions in Orlicz spaces, some partial ordering relations were treated in Chapter II of [10]. In [9] Mazzone and Zó introduce the quasi-increasing function's concept, they define the relation $\prec$ between two non negative functions and they determine some properties of the relation. Later, in [1], it is defined and thoroughly studied another relation $\prec_N$. Both relations are used to obtain strong type inequalities as follows.

Let $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a non decreasing function such that $\varphi(0) = 0$ and satisfies a weak type inequality like

$$\mu(\{f > a\}) \leq C_w \int_{\{f>a\}} \frac{\varphi(g)}{\varphi(a)} d\mu \quad \text{for all } \ a > 0,$$

or an extra-weak type inequality like

$$\mu(\{f > a\}) \leq 2C_w \int_{\{f>a\}} \varphi\left(\frac{g}{a}\right) d\mu \quad \text{for all } \ a > 0,$$

where $f,g : \Omega \to \mathbb{R}_0^+$ are two fixed measurable functions. Then, in [9] and [1] it has considered functions $\Psi \in C^1([0,\infty))$, $\Psi(x) = \int_0^x \psi(t)dt$ and $\varphi \prec \psi$ or $\varphi \prec_N \psi$, which allows us to get strong type inequalities like

$$\int_\Omega \Psi(f)d\mu \leq 2C_w\rho \int_\Omega \Psi\left(\frac{2}{c}g\right) d\mu. \tag{1.1}$$