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Nearly Comonotone Approximation of Periodic Functions

G. A. Dzyubenko*

Yu. A. Mitropolsky International Mathematical Center of NASU, 01004 Kyiv, 3, Tereschenkivska st., Ukraine

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Abstract. Suppose that a continuous 2π-periodic function f on the real axis changes its monotonicity at points $y_i : -\pi \le y_{2s} < y_{2s-1} < \cdots < y_1 < \pi$, $s \in \mathbb{N}$. In this paper, for each $n \ge N$, a trigonometric polynomial P_n of order cn is found such that: P_n has the same monotonicity as f, everywhere except, perhaps, the small intervals

$$(y_i - \pi/n, y_i + \pi/n)$$

and

$$\|f - P_n\| \le c(s)\omega_3(f, \pi/n),$$

where N is a constant depending only on $\min\limits_{i=1,\cdots,2s}\{y_i - y_{i+1}\}$, c, $c(s)$ are constants depending only on s, $\omega_3(f,\cdot)$ is the modulus of smoothness of the 3-rd order of the function f, and $\|\cdot\|$ is the max-norm.

Key Words: Periodic functions, comonotone approximation, trigonometric polynomials, Jackson-type estimates.

AMS Subject Classifications: 41A10, 41A17, 41A25, 41A29

1 Introduction and the main theorem

By C we denote the space of continuous 2π-periodic functions $f : \mathbb{R} \to \mathbb{R}$ with the uniform norm

$$\|f\| := \|f\|_{\mathbb{R}} = \max_{x \in \mathbb{R}} |f(x)|,$$

and by $\mathbb{T}_n, n \in \mathbb{N}$, denote the space of trigonometric polynomials

$$P_n(x) = a_0 + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx), \quad a_j \in \mathbb{R}, \quad b_j \in \mathbb{R},$$

*Corresponding author. *Email address:* `dzyuben@gmail.com` (G. A. Dzyubenko)

of degree $\leq n$. Recall the classical Jackson-Zygmund-Akhiezer-Stechkin estimate (obtained by Jackson for $k=1$, Zygmund and Akhiezer for $k=2$, and Stechkin for $k\geq 3$, $k\in\mathbb{N}$): *if a function $f\in C$, then for each $n\in\mathbb{N}$ there is a polynomial $P_n\in\mathbb{T}_n$ such that*

$$\|f-P_n\|\leq c(k)\omega_k(f,\pi/n), \tag{1.1}$$

where $c(k)$ is a constant depending only on k, and $\omega_k(f,\cdot)$ is the modulus of continuity of order k of the function f. For details, see, for example, [2].

In 1968 Lorentz and Zeller [7] for $k=1$ obtained a bell-shaped analogue of the inequality (1.1), i.e., when bell-shaped (even and nonincreasing on $[0,\pi]$) 2π-periodic functions are approximated by bell-shaped polynomials.

In papers [9] and [4] a comonotone analogue of the inequality (1.1) was proved for $k=1$ and $k=2$, respectively. Moreover, in [8] arguments from the papers [12,13] of Shvedov and [1] of DeVore, Leviatan and Shevchuk were used to show that for $k>2$ there is no comonotone analogue of the inequality (1.1).

Nevertheless, as we know from the comonotone approximation on a closed interval (by algebraic polynomials, see, for details [5]) *if some relaxation of the condition of comonotonicity for approximating polynomials is allowed, then an extra order of the approximation can be achieved, and no more than one extra order*, see the corresponding counterexample in [6].

So, in this paper in Theorem 1.1 we prove a trigonometric analogue of this algebraic result by Leviatan and Shevchuk [5]. To write it we give necessary notations.

Suppose that on $[-\pi,\pi)$ there are $2s$, $s\in\mathbb{N}$, fixed points y_i:

$$-\pi\leq y_{2s}<y_{2s-1}<\cdots<y_1<\pi,$$

while for other indices $i\in\mathbb{Z}$, the points y_i are defined periodically by the equality

$$y_i=y_{i+2s}+2\pi \quad (\text{i.e., } y_0=y_{2s}+2\pi,\cdots,y_{2s+1}=y_1-2\pi,\cdots).$$

Denote $Y:=\{y_i\}_{i\in\mathbb{Z}}$. By $\Delta^{(1)}(Y)$ we denote the set of all functions $f\in C$ which are nondecreasing on $[y_1,y_0]$, nonincreasing on $[y_2,y_1]$, nondecreasing on $[y_3,y_2]$, and so on. The functions in $\Delta^{(1)}(Y_s)$ are *comonotone* with one another. Note, if a function f is differentiable, then $f\in\Delta^{(1)}(Y)$ if and only if

$$f'(x)\Pi(x)\geq 0, \quad x\in\mathbb{R},$$

where

$$\Pi(x):=\Pi(x,Y):=\prod_{i=1}^{2s}\sin\frac{x-y_i}{2}, \qquad (\Pi(x)>0, \quad x\in(y_1,y_0)).$$

Theorem 1.1. *If a function $f\in\Delta^{(1)}(Y)$, then there exists a constant $N(Y)$ depending only on $\min\limits_{i=1,\cdots,2s}\{y_i-y_{i+1}\}$ such that for each $n\geq N(Y)$ there is a polynomial $P_n\in\mathbb{T}_{cn}$ satisfying*

$$P_n'(x)\Pi(x)\geq 0, \quad x\in\mathbb{R}\setminus\cup_{i\in\mathbb{Z}}(y_i-\pi/n,y_i+\pi/n), \tag{1.2a}$$
$$\|f-P_n\|\leq c(s)\omega_3(f,\pi/n), \tag{1.2b}$$

where c, $c(s)$ are constant depending only on s.