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On the Green Function of the Annulus

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Abstract. Using the Gegenbauer polynomials and the zonal harmonics functions we give some representation formula of the Green function in the annulus. We apply this result to prove some uniqueness results for some nonlinear elliptic problems.

Key Words: Green's function, symmetries, uniqueness.

AMS Subject Classifications: 35B09

1 Introduction and statement of the main results

The classical Green function of the operator $-\Delta$ with Dirichlet boundary conditions is defined by

$$\begin{cases} -\Delta_x G(x,y)=\delta_y(x) & \text{in } \Omega, \\ G(x,y)=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where δ_y is the Dirac function centered at y and Ω is a bounded domain of $\mathbb{R}^n$, $n\geq 2$.

It is well known that the Green function can be written as

$$G(x,y)=\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}+H(x,y), \tag{1.2}$$

where $H(x,y)$ is a smooth function in $\Omega\times\Omega$ which is harmonic in both variables x and y. Finally the Robin function is defined as

$$R(x)=H(x,x). \tag{1.3}$$

The knowledge of the Green (or the Robin) function is of great importance in applications (we mention the paper [2] and the rich list of references therein). Indeed the explicit

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calculation of the Green function is an old problem (see for example the book by Courant and Hilbert, [5]) but it can be solved only in special cases (like the ball or half-space).

For these reason, even if it is not possible to have the explicit expression, it is very important to deduce any properties of the Green function.

In this paper we are interested to the case where the domain is the *annulus* in $\mathbb{R}^n$, namely $\Omega=\{x\in\mathbb{R}^n:\ a<|x|<b\}$ (in the rest of the paper by simplicity we assume that $b=1$). Even if the annuls possesses many symmetries, you can not explicitly write the Green function. If $n=2$ in [7] it was given a representation for the Green function as trigonometrical series. In this paper we give a representation formula of the Green function when $n\geq 3$ using the *zonal spherical harmonics*. Our first result is the following,

Theorem 1.1. *Let A be the annulus $A=\{x\in\mathbb{R}^n:\ a<|x|<1\}$. Then we have that the Green function in A is given by,*

$$\begin{aligned}&G_A(x,y)\\=&\frac{1}{(n-2)\omega_{n-1}|x-y|^{n-2}}\\&-\frac{1}{\omega_{n-1}}\sum_{m=0}^{\infty}\frac{|x|^{2m+n-2}|y|^{2m+n-2}-a^{2m+n-2}(|x|^{2m+n-2}+|y|^{2m+n-2})+a^{2m+n-2}}{(2m+n-2)(|x||y|)^{n+m-2}(1-a^{2m+n-2})}Z_m\Big(\frac{x}{|x|},\frac{y}{|y|}\Big).\end{aligned}\tag{1.4}$$

Moreover the Robin function is given by, setting $d_0=1$ and

$$d_m=\binom{n+m-2}{n-2}-\binom{n+m-3}{n-2,},$$

for $m\geq 1$,

$$R_A(x)=-\frac{1}{\omega_{n-1}}\sum_{m=0}^{\infty}d_m\frac{a^{2m+n-2}-2a^{2m+n-2}|x|^{2m+n-2}+|x|^{4m+2n-4}}{(2m+n-2)|x|^{2m+2n-4}(1-a^{2m+n-2})}.\tag{1.5}$$

Here $Z_m(x,y)$ are the zonal spherical harmonics (see Section 2 or [1] for the definition and main properties).

Next corollary gives an alternative expression of the Green function which does not involve the Newtonian potential.

Corollary 1.1. Let A be the annulus $A=\{x\in\mathbb{R}^n: a<|x|<1\}$. Then we have that the Green function is given by,

$$G_A(x,y)=\begin{cases}\dfrac{1}{\omega_{n-1}}\displaystyle\sum_{m=0}^{\infty}\frac{(|x|^{2m+n-2}-a^{2m+n-2})(1-|y|^{2m+n-2})}{(2m+n-2)(|x||y|)^{n+m-2}(1-a^{2m+n-2})}Z_m\Big(\frac{x}{|x|},\frac{y}{|y|}\Big), & \text{if } |x|<|y|,\\ \dfrac{1}{2^{n-2}|x|^{n-2}|1-\frac{x\cdot y}{|x|^2}|}+R_A(x), & \text{if } |x|=|y|,\quad x\neq y,\\ \dfrac{1}{\omega_{n-1}}\displaystyle\sum_{m=0}^{\infty}\frac{(|y|^{2m+n-2}-a^{2m+n-2})(1-|x|^{2m+n-2})}{(2m+n-2)(|x||y|)^{n+m-2}(1-a^{2m+n-2})}Z_m\Big(\frac{x}{|x|},\frac{y}{|y|}\Big), & \text{if } |x|>|y|.\end{cases}\tag{1.6}$$