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H^1-Estimates of the Littlewood-Paley and Lusin Functions for Jacobi Analysis II

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Abstract. Let $(\mathbb{R}_+,*,\Delta)$ be the Jacobi hypergroup. We introduce analogues of the Littlewood-Paley g function and the Lusin area function for the Jacobi hypergroup and consider their (H^1,L^1) boundedness. Although the g operator for $(\mathbb{R}_+,*,\Delta)$ possesses better property than the classical g operator, the Lusin area operator has an obstacle arisen from a second convolution. Hence, in order to obtain the (H^1,L^1) estimate for the Lusin area operator, a slight modification in its form is required.

Key Words: Jacobi analysis, Jacobi hypergroup, g function, area function, real Hardy space.

AMS Subject Classifications: 22E30, 43A30, 43A80

1 Introduction

One of main subjects of the so-called real method in classical harmonic analysis related to the Poisson integral $f*p_t$ is to investigate the Littlewood-Paley theory. For example, in the one dimensional setting, the following singular integral operators were respectively well-known as the Littlewod-Paley g function and the Lusin area function

$$g^{\mathbb{R}}(f)(x)=\Big(\int_0^\infty\Big|f*t\frac{\partial}{\partial t}p_t(x)\Big|^2\frac{dt}{t}\Big)^{1/2}, \tag{1.1a}$$

$$S^{\mathbb{R}}(f)(x)=\Big(\int_0^\infty\frac{1}{t}\chi_t*\Big|f*t\frac{\partial}{\partial t}p_t\Big|^2(x)\frac{dt}{t}\Big)^{1/2}, \tag{1.1b}$$

where χ_t is the characteristic function of $[-t,t]$. These operators satisfy the maximal theorem, that is, a weak type L^1 estimate and a strong type L^p estimate for $1<p\le\infty$. Moreover, they are bounded form H^1 into L^1 (cf. [10–12]). Our matter of concern is to extend these results to other topological spaces X. Roughly speaking, in some examples of X of homogeneous type (see [2]), Poisson integrals are generalized on X and analogous

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Littlewood-Paley theory has been developed (cf. [2,5,10]). On the other hand, if the space X is not of homogeneous type, we encounter difficulties. As an example of X of non homogeneous type with Poisson integrals, noncompact Riemannian symmetric spaces $X=G/K$ are well-known. Lohoue [9] and Anker [1] generalize the Littlewood-Paley g function and the Luzin area function to G/K and show that they satisfy the maximal theorem (see below). However, we know little or nothing whether they are bounded from H^1 into L^1, because we first have to find out a suitable definition of a real Hardy space on G/K. The aim of this paper is to introduce a real Hardy space $H^1(\Delta)$ and show that they are bounded from $H^1(\Delta)$ into $L^1(\Delta)$ for the Jacobi hypergroup $(\mathbb{R}_+,*,\Delta)$, which is a generalization of K-invariant setting on G/K of real rank one.

We briefly overview the Jacobi hypergroup $(\mathbb{R}_+,*,\Delta)$. We refer to [4] and [8] for a description of general context. For $\alpha\geq\beta\geq-\frac{1}{2}$ and $(\alpha,\beta)\neq(-\frac{1}{2},-\frac{1}{2})$ we define the weight function Δ on $\mathbf{R}_+$ as

$$\Delta(x)=(2\mathrm{sh}x)^{2\alpha+1}(2\mathrm{ch}x)^{2\beta+1}.$$

Clearly, it follows that

$$\Delta(x)\leq c\begin{cases} e^{2\rho x}, & x>1,\\ x^{2\gamma_0}, & x\leq 1,\end{cases}$$

where $\rho=\alpha+\beta+1$ and $\gamma_0=\alpha+\frac{1}{2}$. For $\lambda\in\mathbb{C}$ let ϕ_λ be the Jacobi function on $\mathbb{R}_+$ defined by

$$\phi_\lambda(x)={}_2F_1\Big(\frac{\rho+i\lambda}{2},\frac{\rho-i\lambda}{2};\alpha+1;-(\mathrm{sh}x)^2\Big),$$

where ${}_2F_1$ the hypergeometric function. Then the Jacobi transform $\hat{f}$ of a function f on $\mathbb{R}_+$ is defined by

$$\hat{f}(\lambda)=\frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx.$$

We define a generalized translation on $\mathbb{R}_+$ by using the kernel form of the product formula of Jacobi functions: For $x,y\in\mathbb{R}_+$,

$$\phi_\lambda(x)\phi_\lambda(y)=\int_0^\infty \phi_\lambda(z)K(x,y,z)\Delta(z)dx.$$

The kernel $K(x,y,z)$ is non-negative and symmetric in the tree variables. Then the generalized translation T_x of f is defined as

$$T_xf(y)=\int_0^\infty f(z)K(x,y,z)\Delta(z)dz$$

and the convolution of f, g is given by

$$f*g(x)=\int_0^\infty f(y)T_xg(y)\Delta(y)dy.$$