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On Copositive Approximation in Spaces of Continuous Functions II: The Uniqueness of Best Copositive Approximation

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Abstract. This paper is part II of "On Copositive Approximation in Spaces of Continuous Functions". In this paper, the author shows that if Q is any compact subset of real numbers, and M is any finite dimensional strict Chebyshev subspace of $C(Q)$, then for any admissible function $f \in C(Q)\backslash M$, the best copositive approximation to f from M is unique.

Key Words: Strict Chebyshev spaces, best copositive approximation, change of sign.

AMS Subject Classifications: 41A65

1 Introduction

If Q is a compact Hausdorff space, then $C(Q)$ denotes the Banach space of all continuous real valued functions on Q, together with the uniform norm, that is, $\|f\| = \max\{|f(x)| : x \in Q\}$. If M is a subspace of $C(Q)$, and $f \in C(Q)$, then $g \in M$ is said to be copositive with f on Q iff $f(x)g(x) \geq 0$ for all $x \in Q$. The element $g_0 \in M$ is called a best copositive approximation to f from M iff g_0 is copositive with f on Q and $\|f - g_0\| = \inf\{\|f-g\| : g \in M$, and g is copositive with$f$ on $Q\}$. The set $\{g \in M : g$ is copositive with f on $Q\}$ is closed, so if the dimension of M is finite, then the best copositive approximation to each $f \in C(Q)$ from M is attained. If Q is a compact subset of real numbers, then the n-dimensional subspace M of $C(Q)$ is called Chebyshev subspace of $C(Q)$ if each $g \neq 0$ in M has at most $n-1$ zeros. The n-dimensional Chebyshev subspace M of $C(Q)$ is called a "Strict Chebyshev subspace" of $C(Q)$ if each $g \neq 0$ in M has at most $n-1$ changes of signs, that is, no $g \neq 0$ in M alternates strongly at $n+1$ points of Q, which means that there do not exist $n+1$ points, $x_0 < x_2 < \cdots < x_{n+1}$ in Q so that $g(x_i)g(x_{i+1}) < 0$ for all $i = 1,2,\cdots,n$.

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This paper is a continuation of the author's paper [1]. In this paper the author investigates the uniqueness of the best copositive approximation by elements of finite dimensional subspaces of $C(Q)$. Passow and Taylor [2] showed that when Q is any finite subset of real numbers, and M is a finite dimensional strict Chebyshev subspace of $C(Q)$ then the best copositive approximation to each $f \in C(Q)$ from M is unique. Zhong [3] proved the same result for the case when Q is a closed and bounded interval $[a,b]$ of the real numbers, and f does not vanish on any subinterval of $[a,b]$. In this paper it is shown that this fact is true for any compact subset of real numbers.

The rest of this section will be used to cover some notation and results that will be used later in Section 2. As in Kamal [1], If Q is a compact subset of real numbers, and $x_1<x_2$ in Q then the "intervals" $(x_1,x_2),(x_1,x_2]$, $[x_1,x_2)$, and $[x_1,x_2]$ in Q are defined in the ordinary way, for example; $(x_1,x_2)=\{x\in Q:x_1<x<x_2\}$. If Q is not connected then none of those intervals need to be connected. The point z_0 in Q is called "a limit point from both sides" in Q if z_0 is an accumulation for the set $\{x\in Q:x<z_0\}$, and the set $\{x\in Q:x>z_0\}$. If z_0 is an accumulation point for the set $\{x\in Q:x<z_0\}$ or the set $\{x\in Q:x>z_0\}$ but not for both then z_0 is called "a limit point from one side" in Q. The function $f\in C(Q)$ is said to have at "least k changes of sign in Q" if there are $k+1$ point $t_1<t_2<\cdots,t_{k+1}$ in Q so that $f(t_i)f(t_{i+1})<0$ for all $i=1,2,\cdots,k$. The "number of changes of sign of f" is defined to be the $\sup\{k:f$ has at least k changes of sign$\}$. Assume that $f\neq 0$ in $C(Q)$, the point $z\in Q$ is said to be a "double zero" for f in Q if $f(z)=0$, and there are $x<z<y$ in Q so that $f(\alpha)f(\beta)>0$ for all $\alpha\neq z$, and $\beta\neq z$ in $[x,y]$. If $f(z)=0$, and z is not a double zero then z is called a "single zero" in Q (see [4]). Finally the function $f\in C(Q)$ is called admissible if f does not vanish on any infinite interval of Q.

The following Proposition presents some of the known propertis of strict Chebyshev subspaces.

Proposition 1.1. Assume that Q is a compact subset of real numbers containing at least $n+1$ points, and that M is an n-dimensional strict Chebyshev subspace of $C(Q)$. The following facts hold;

i). If $z_1<z_2<\cdots<z_{n-1}$ are $n-1$ points in Q, then there is $g\in M$, such that $g(x)=0$ for all $x\in\{z_1,z_2,\cdots,z_{n-1}\}$ and;

1). $g(x)>0$, if $x<z_1$,

2). $(-1)^{n-1}g(x)>0$ if $x>z_{n-1}$, and;

3). $(-1)^i g(x)>0$ if $x\in(z_i,z_{i+1})$, and $i=1,2,\cdots,n-1$.

ii). No $g\neq 0$ in M alternates weakly at $n+1$ points in Q, that is, there do not exist $x_1<x_2<\cdots<x_{n+1}$ in Q, and $g\neq 0$ in M such that $(-1)^i g(x_i)\geq 0$ for each $i=1,2,\cdots,n+1$.

iii). If $g\neq 0$ in M and k is the number of single zeros of g, and m is the number of double zeros of g then $k+2m\leq n-1$.

Part i) in Proposition 1.1 can be obtained from Lemma 6.5 in Zielke [4], part ii), is in [4, Lemma 3.1b], part iii) is [4, Lemma 6.2].