On an Inequality of Paul Turan Concerning Polynomials-II

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Abstract. Let \( P(z) \) be a polynomial of degree \( n \) and for any complex number \( \alpha \), let 
\[ D_\alpha P(z) = nP(z) + (\alpha - z)P'(z) \]
denote the polar derivative of the polynomial \( P(z) \) with respect to \( \alpha \). In this paper, we obtain inequalities for the polar derivative of a polynomial having all zeros inside a circle. Our results shall generalize and sharpen some well-known results of Turan, Govil, Dewan et al. and others.

Key Words: Polar derivative, polynomials, inequalities, maximum modulus, growth.

AMS Subject Classifications: 30A10, 30C10, 30C15

1 Introduction and statement of results

Let \( P(z) \) be a polynomial of degree \( n \) and \( P'(z) \) be its derivative. Then according to the well-known Bernstein’s inequality [4] on the derivative of a polynomial, we have

\[
\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1.1}
\]

The equality holds in (1.1) if and only if \( P(z) \) has all its zeros at the origin.

For the class of polynomials \( P(z) \) having all zeros in \( |z| \leq 1 \), Turan [11] proved that

\[
\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}
\]

The inequality (1.2) is best possible and becomes equality for \( P(z) = \alpha z^n + \beta \) where \( |\alpha| = |\beta| \).

In the literature, there already exists some refinements and generalizations of the inequality (1.2), for example see Aziz and Dawood [3], Govil [5], Dewan and Mir [6], Dewan, Singh and Mir [7], Mir, Dar and Dawood [10] etc.

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Inequality (1.2) was refined by Aziz and Dawood [3] and they proved under the same hypothesis that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\}. \quad (1.3)$$

As an extension of (1.3), it was shown by Govil [5], that if $P(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k} \left\{ \max_{|z|=1} |P(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |P(z)| \right\}. \quad (1.4)$$

For the class of polynomials

$$P(z) = a_n z^n + \sum_{\nu=0}^{n-\mu} a_{n-\nu} z^{n-\nu}, \quad 1 \leq \mu \leq n,$$

of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, Aziz and Shah [2] proved

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k\mu} \left\{ \max_{|z|=1} |P(z)| + \frac{1}{k^{n-1}} \min_{|z|=k} |P(z)| \right\}. \quad (1.5)$$

For $\mu = 1$, inequality (1.5) reduces to (1.4).

Let $D_\alpha P(z)$ denote the polar derivative of the polynomial $P(z)$ of degree $n$ with respect to $\alpha$, then

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Recently Dewan, Singh and Mir [7] besides proving some other results, also proved the following interesting generalization of (1.5).

**Theorem 1.1.** If

$$P(z) = a_n z^n + \sum_{\nu=0}^{n-\mu} a_{n-\nu} z^{n-\nu}, \quad 1 \leq \mu \leq n,$$

is a polynomial of degree $n$ having all its zeros in $|z| \leq k$, $k \leq 1$, and $\delta$ is any complex number with $|\delta| \leq 1$, then for $|z| = 1$,

$$|D_\delta P(z)| \leq n \left( \frac{k^\mu + |\delta|}{1+k^n} \right) \max_{|z|=1} |P(z)| - n \left( \frac{1-|\delta|}{k^{n-\mu}(1+k^n)} \right) \min_{|z|=k} |P(z)|. \quad (1.6)$$

In this paper, we shall first prove a result which gives certain generalizations of the inequality (1.4) by considering polynomials having all zeros in $|z| \leq k$, $k \leq 1$ with $s$-fold zeros at $z = 0$. We shall also present a refinement of Theorem 1.1. We first prove the following result.