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Some Remarks on the Restriction Theorems for the Maximal Operators on $\mathbb{R}^d$

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Abstract. The aim of this paper is to give a simple proof of the restriction theorem for the maximal operators on the d-dimensional Euclidean space $\mathbb{R}^d$, whose theorem was proved by Carro-Rodriguez in 2012. Moreover, we shall give some remarks of the restriction theorem for the linear and the multilinear operators by Carro-Rodriguez and Rodriguez, too.

Key Words: Weighted L^p spaces, Fourier multipliers, multilinear operators.

AMS Subject Classifications: 42B15,42B35

1 Introduction and results

Let p be in $1\le p<\infty$, $w(x)$ a nonnegative 2π periodic function in $L^1_{loc}(\mathbb{R}^d)$ which is called a weight. First we define weighted L^p spaces on the d-dimensional Euclidean space $\mathbb{R}^d$ or on the d-dimensional torus $\mathbb{T}^d$.

Definition 1.1. Let $1\le p<\infty$, $0<q<\infty$, and $w(x)$ a non-negative 2π periodic function in $L^1_{loc}(\mathbb{R}^d)$

$$L^{p,q}(\mathbb{R}^d,w)=\Big\{f\,|\,\|f\|_{L^{p,q}(\mathbb{R}^d,w)}=\Big(\int_0^\infty (tw(\{|f|>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}<\infty\Big\},$$
$$L^{p,\infty}(\mathbb{R}^d,w)=\Big\{f\,|\,\|f\|_{L^{p,\infty}(\mathbb{R}^d,w)}=\inf\big\{M\,|\,tw(\{x\in\mathbb{R}^d\,\|f(x)|>t\})^{1/p}<M \ \text{ for } \ t>0\big\}<\infty\Big\},$$
$$L^{p,q}(\mathbb{T}^d,w)=\Big\{F\,|\,\|F\|_{L^{p,q}(\mathbb{R}^d,w)}=\Big(\int_0^\infty (tw(\{|F|>t\})^{1/p})^q\frac{dt}{t}\Big)^{1/q}<\infty\Big\},$$
$$L^{p,\infty}(\mathbb{T}^d,w)=\Big\{F\,|\,\|F\|_{L^{p,\infty}(\mathbb{T}^d,w)}=\inf\big\{M\,|\,tw(\{x\in\mathbb{R}^d\,\|F(x)|>t\})^{1/p}<M \ \text{ for } \ t>0\big\}<\infty\Big\},$$

where $w(E)=\int_E w(x)dx$ for $E\subset\mathbb{R}^d$ or $E\subset\mathbb{T}^d$.

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Also let $\{\phi_j\}_{j=1}^{\infty}$ be in $C_b(\mathbb{R}^d)$ which is the set of all bounded continuous functions on $\mathbb{R}^d$, and $\phi_{j|_{\mathbb{Z}^d}}$ the restriction function of ϕ_j on the d-dimensional integer space $\mathbb{Z}^d$. When $w(x)=1$ $(x\in\mathbb{R}^d)$, $L^p(\mathbb{R}^d,w)$, $L^{p,\infty}(\mathbb{R}^d,w)$ (resp. $L^p(\mathbb{T}^d,w)$, $L^{p,\infty}(\mathbb{T}^d,w)$) are denoted by $L^p(\mathbb{R}^d)$, $L^{p,\infty}(\mathbb{R}^d)$ (resp. $L^p(\mathbb{T}^d)$, $L^{p,\infty}(\mathbb{T}^d)$), respectively. Moreover, we define some operators T_{ϕ_j}, T^*, $\widetilde{T_{\phi_j|_{\mathbb{Z}^d}}}$, and $\widetilde{T^*}$:

Definition 1.2. For

$$T_{\phi_j}f(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\phi_j(\xi)\hat{f}(\xi)e^{ix\xi}d\xi, \qquad T^*f(x)=\sup_j|T_{\phi_j}f(x)|,$$

$$\widetilde{T_{\phi_{j|\mathbb{Z}^d}}}F(x)=\sum_{k\in\mathbb{Z}^d}\phi_j(k)\hat{F}(k)e^{ikx}, \qquad \widetilde{T^*}F(x)=\sup_j|\widetilde{T_{\phi_j|\mathbb{Z}^d}}F(x)|,$$

where f is in Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$, and F in trigonometric polynomials $P(\mathbb{T}^d)$ on $\mathbb{T}^d$,

$$\hat{f}(\xi)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}f(x)e^{-i\xi x}dx \text{ and } \widehat{F}(k)=\frac{1}{(2\pi)^d}\int_{[0,2\pi)^d}F(x)e^{-ikx}dx\Big(=\int_{\mathbb{T}^d}F(x)e^{-ikx}dx\Big).$$

Now in 1960, K. de Leeuw [5] proved that if T_ϕ is bounded on $L^p(\mathbb{R}^d)$ for $\phi\in C_b(\mathbb{R}^d)$, $\widetilde{T_{\phi_{|\mathbb{Z}^d}}}$ is bounded on $L^p(\mathbb{T}^d)$. In 1985, Kenig-Tomas [14] showed the same result between T^* and $\widetilde{T^*}$ for $1<p<\infty$. Moreover, in 1994, Asmar-Berkson-Bourgain [2] (cf. [1,12]) proved that if T^* is bounded from $L^p(\mathbb{R}^d)$ to $L^{p,\infty}(\mathbb{R}^d)$, $\widetilde{T^*}$ is bounded from $L^p(\mathbb{T}^d)$ to $L^{p,\infty}(\mathbb{T}^d)$ for $1\le p<\infty$. After that, there are many papers related to this property [6,7] (cf. [8,9,17]). Also in 2003, Berkson-Gillispie [3] proved that if T_ϕ is bounded on $L^p(\mathbb{R}^d,w)$ for $\phi\in C_b(\mathbb{R}^d)$ and $1<p<\infty$ with $w\in A_p(\mathbb{T}^d)$, $\widetilde{T_{\phi|Z^d}}$ is bounded on $L^p(\mathbb{T}^d,w)$, where

$$A_p(\mathbb{T}^d)=\Big\{w(x)\ge 0|w(x) \text{ is a } 2\pi \text{ periodic function on } \mathbb{R}^d$$
$$\text{with } \sup_{Q,\text{cube}}\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\Big)^{p-1}<\infty\Big\},$$

where $|Q|$ is the Lebesgue measure of $Q\subset\mathbb{T}^d$. In 2009, Anderson-Mohanty [1] proved Berkson-Gillispie's result without A_p condition. In 2012, Carro-Rodriguez [4] which is summing up to the restriction theorems of multipliers in weighted setting showed that if T^* is bounded from $L^p(\mathbb{R}^d,w)$ to $L^{p,\infty}(\mathbb{R}^d,w)$ for $1\le p<\infty$ with a non-negative 2π periodic function $w(x)\in L^1_{loc}(\mathbb{R}^d)$, $\widetilde{T^*}$ is bounded from $L^p(\mathbb{T}^d,w)$ to $L^{p,\infty}(\mathbb{T}^d,w)$ (cf. [13]). Their results are proved by applying Kolmogorov's condition with vector valued argument (cf. [10]).

Recently by the same method, Rodriguez [15] gives the analogy with respect to the multilinear operators, whose result is as follows: Let $1\le p_j<\infty$ $(j=1,\cdots,m)$ for $m\in\mathbb{N}$ with $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$, and $w(x),w_1(x),\cdots,w_m(x)$ 2π periodic non-negative functions. Also let