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On Weighted Approximation by Modified Bernstein Operators for Functions with Singularities

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Abstract. Della Vecchia et al. (see [2]) introduced a kind of modified Bernstein operators which can be used to approximate functions with singularities at endpoints on [0,1]. In the present paper, we obtain a kind of pointwise Stechkin-type inequalities for weighted approximation by the modified Bernsetin operators.

Key Words: Modified Bernstein operators, weighted approximation, Stechkin inequality.

AMS Subject Classifications: 41A28, 41A36

1 Introduction

Let $C_{[0,1]}$ be the set of all continuous functions on [0,1]. For any $f\in C_{[0,1]}$, the corresponding Bernstein operators are defined by

$$B_n(f,x)=\sum_{k=0}^{n} p_{n,k}(x)f\Big(\frac{k}{n}\Big),\quad f\in C_{[0,1]},$$

where

$$p_{n,k}(x):=\binom{n}{k}x^k(1-x)^{n-k},\quad x\in[0,1],\quad k=0,1,\cdots,n.$$

Set

$$\Delta^2_{h\varphi^\lambda}f(x):=\begin{cases} f(x+h\varphi^\lambda(x))-2f(x)+f(x-h\varphi^\lambda(x)), & x\pm h\varphi^\lambda(x)\in[0,1],\\ 0, & \text{otherwise,}\end{cases}$$

where $\varphi(x)=\sqrt{x(1-x)}$. Denote by $\|f\|_{[a,b]}$ the uniform norm of $f\in C_{[a,b]}$. Especially, write $\|f\|:=\|f\|_{[0,1]}$. For the direct results, Ditzian (see [4]) obtained the following classical estimates:

$$|B_n(f,x)-f(x)|\le C\omega^2_{\varphi^\lambda}\Big(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big), \tag{1.1}$$

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where

$$\omega^2_{\varphi^\lambda}(f,t):=\sup_{0<h\le t}\|\Delta^2_{h\varphi^\lambda}f\|,\quad 0\le\lambda\le 1.$$

As we know, (1.1) contains both the global estimate $(\lambda=0)$ and the ponitwise estimates $(0\le\lambda<1)$.

For the converse result, Wickeren (see [9]) established the following Stechkin-Marchaud type inequality:

$$\omega^2_\alpha\Big(f,\frac{1}{\sqrt{n}}\Big)\le C\frac{1}{n}\sum_{k=1}^n\|\varphi^{-\alpha}(B_k(f)-f)\|,\quad 0\le\alpha\le 2,$$

where

$$\omega^2_\alpha(f,t):=\sup_{0<h\le t}\{\varphi^{-\alpha}|\Delta^2_{h\varphi(x)}f(x)|:\ x,\ x\pm h\varphi(x)\in[0,1]\}.$$

The approximation of continuous functions by Bernstein operators under uniform norm is studied very extensively. However, there are still many problems on the weighted approximation by Bernstein operators. As we know, there are some essential differences between uniform approximation and weighted approximation. For example, Bernstein operators is not bounded under weighted norm with Jacobi weights $w(x)=x^a(1-x)^b$, $0<a,b<1$ (see [13]). To overcome the difficulty, Zhou (see [13]) introduced the following new weighted norm:

$$\|f\|_w:=\sup_{x\in(0,1)}|w(x)f(x)|+|f(0)|+|f(1)|. \tag{1.2}$$

Under this new weighted norm, Bernstein operators are bounded. Later, some authors investigated the approximation properties of Bernstein operators under the norm (1.2). Among others, Wang (see [7]) obtained the following Stechkin-Marchaud type inequality:

$$\Omega^2_{\varphi^\lambda}\Big(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big)_w\le C\frac{1}{n}\sum_{k=1}^n\|B_k(f)-f\|_0,\quad f(x)\in C_{\lambda,\alpha,\beta}, \tag{1.3}$$

where

$$\begin{aligned}
&\Omega^2_{\varphi^\lambda}(f,t)_\omega:=\sup_{0<h\le t}\Big\{|w(x)\varphi^{\alpha(\lambda-1)-\beta}(x)\Delta^2_{h\varphi^\lambda}f(x)|,\ x\pm h\varphi(x)\in[0,1]\Big\},\\
&\|f\|_0:=\sup_{x\in(0,1)}|w(x)\varphi^{\alpha(\lambda-1)-\beta}(x)f(x)|,\quad 0<\alpha<2,\quad 0\le\beta\le 2,\\
&C_{\lambda,\alpha,\beta}:=\{f:f\in C[0,1],\ \|f\|_0<\infty\}.
\end{aligned}$$

(1.3) generalized the related result of [9].