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Some Generalization for an Operator Which Preserving Inequalities Between Polynomials

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Abstract. For a polynomial $p(z)$ of degree n which has no zeros in $|z|<1$, Dewan et al., (K. K. Dewan and Sunil Hans, Generalization of certain well known polynomial inequalities, J. Math. Anal. Appl., 363 (2010), 38–41) established

$$\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\le\frac{n}{2}\Big\{\Big(\Big|\frac{\beta}{2}\Big|+\Big|1+\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|p(z)|-\Big(\Big|1+\frac{\beta}{2}\Big|-\Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|p(z)|\Big\},$$

for any complex number β with $|\beta|\le 1$ and $|z|=1$. In this paper we consider the operator B, which carries a polynomial $p(z)$ into

$$B[p(z)]:=\lambda_0 p(z)+\lambda_1\Big(\frac{nz}{2}\Big)\frac{p'(z)}{1!}+\lambda_2\Big(\frac{nz}{2}\Big)^2\frac{p''(z)}{2!},$$

where λ_0, λ_1, and λ_2 are such that all the zeros of $u(z)=\lambda_0+c(n,1)\lambda_1 z+c(n,2)\lambda_2 z^2$ lie in the half plane $|z|\le|z-n/2|$. By using the operator B, we present a generalization of result of Dewan. Our result generalizes certain well-known polynomial inequalities.

Key Words: B-operator, inequality, polynomial, maximum modulus, restricted zeros.

AMS Subject Classifications: 30A10, 30C10, 30D15

1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree n and $p'(z)$ its derivative. Then it is well known that

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|, \tag{1.1}$$

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and

$$\max_{|z|=R>1}|p(z)|\le R^n\max_{|z|=1}|p(z)|. \tag{1.2}$$

Inequality (1.1) is a famous result due to Bernstein [7], whereas inequality (1.2) is a simple consequence of maximum modulus principle (see [16]). Both the above inequalities are sharp and equality in each holds for the polynomials having all its zeros at the origin.

For the class of polynomials having no zeros in $|z|<1$, inequalities (1.1) and (1.2) have respectively been replaced by

$$\max_{|z|=1}|p'(z)|\le\frac{n}{2}\max_{|z|=1}|p(z)|, \tag{1.3}$$

and

$$\max_{|z|=R>1}|p(z)|\le\frac{R^n+1}{2}\max_{|z|=1}|p(z)|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erd*ö*s and later proved by Lax [13], whereas inequality (1.4) was proved by Ankeny and Rivlin [1], for which they made use of (1.3). Both these inequalities are also sharp and equality in each holds for polynomials having all its zeros on $|z|=1$.

Aziz and Dawood [4] used $\min_{|z|=1}|p(z)|$ to obtain a refinement of inequalities (1.3) and (1.4) by demonstrating if $p(z)$ is a polynomial of degree n which does not vanish in $|z|<1$, then

$$\max_{|z|=1}|p'(z)|\le\frac{n}{2}\Big\{\max_{|z|=1}|p(z)|-\min_{|z|=1}|p(z)|\Big\}, \tag{1.5}$$

and

$$\max_{|z|=R>1}|p(z)|\le\Big(\frac{R^n+1}{2}\Big)\max_{|z|=1}|p(z)|-\Big(\frac{R^n-1}{2}\Big)\min_{|z|=1}|p(z)|. \tag{1.6}$$

Both these inequalities are also sharp and equality in each holds for polynomials having all its zeros on $|z|=1$.

As refinement of inequalities (1.5) and (1.6), Dewan et al. [8,9] proved that under the same hypothesis, for every $|\beta|\le 1$, $R>1$ and $|z|=1$ we have

$$\Big|zp'(z)+\frac{n\beta}{2}p(z)\Big|\le\frac{n}{2}\Big\{\Big(\Big|1+\frac{\beta}{2}\Big|+\Big|\frac{\beta}{2}\Big|\Big)\max_{|z|=1}|p(z)|-\Big(\Big|1+\frac{\beta}{2}\Big|-\Big|\frac{\beta}{2}\Big|\Big)\min_{|z|=1}|p(z)|\Big\}, \tag{1.7}$$

and

$$\begin{aligned}\Big|p(Rz)+\beta\Big(\frac{R+1}{2}\Big)^n p(z)\Big|\le&\frac{1}{2}\Big\{\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|+\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\max_{|z|=1}|p(z)|\\&-\Big(\Big|R^n+\beta\Big(\frac{R+1}{2}\Big)^n\Big|-\Big|1+\beta\Big(\frac{R+1}{2}\Big)^n\Big|\Big)\min_{|z|=1}|p(z)|\Big\}.\end{aligned} \tag{1.8}$$