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Approximation of Generalized Bernstein Operators

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Abstract. This paper is devoted to study direct and converse approximation theorems of the generalized Bernstein operators $C_n(f,s_n,x)$ via so-called unified modulus $\omega^2_{\varphi^\lambda}(f,t)$, $0\le\lambda\le 1$. We obtain main results as follows

$$\omega^2_{\varphi^\lambda}(f,t)=O(t^\alpha)\Longleftrightarrow|C_n(f,s_n,x)-f(x)|=\mathcal{O}\big((n^{-\frac{1}{2}}\delta_n^{1-\lambda}(x))^\alpha\big),$$

where $\delta_n^2(x)=\max\{\varphi^2(x),1/n\}$ and $0<\alpha<2$.

Key Words: Bernstein type operator, Ditzian-Totik modulus, direct and converse approximation theorem.

AMS Subject Classifications: 41A25, 41A27, 41A36

1 Introduction

Let $C(I)$ be the class of all continuous functions defined on $I=[0,1]$. A generalized Bernstein operator first introduced in [1] is defined by

$$C_n(f,s_n,x)=\frac{1}{s_n}\sum_{i=0}^{n}\sum_{j=0}^{s_n-1}f\Big(\frac{i+j}{n+s_n-1}\Big)p_{n,i}(x),\quad x\in I, \tag{1.1}$$

where

$$p_{n,i}(x)=\binom{n}{i}x^i(1-x)^{n-i},\quad f(x)\in C(I),$$

and $\{s_n\}$ is a sequence of natural numbers.

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Obviously, if $s_n=1$ $(n=1,2,\cdots)$, then $C_n(f,s_n,x)$ degenerates into the well-known Bernstein operators

$$B_nf(x)=\sum_{k=0}^{n}f\Big(\frac{k}{n}\Big)b_{nk}(x),\quad b_{nk}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.2}$$

for a given $f(x)$ on I.

For Bernstein operators, Ditzian has established the following direct theorem of approximation in [2]

$$|B_nf(x)-f(x)|\le C\omega^2_{\varphi^\lambda}\Big(f,\frac{\varphi^{1-\lambda}(x)}{\sqrt{n}}\Big), \tag{1.3}$$

where $\omega^2_{\varphi^\lambda}(f,t)$ is the unified modulus which will be defined in the next section.

When $\lambda=0$, (1.3) degenerates

$$|B_nf(x)-f(x)|\le C\omega^2\Big(f,\frac{\varphi(x)}{\sqrt{n}}\Big),$$

which is a pointwise approximation result; and when $\lambda=1$, (1.3) degenerates

$$|B_nf(x)-f(x)|\le C\omega^2_{\varphi}\Big(f,\frac{1}{\sqrt{n}}\Big),$$

which is a global approximation result. Since (1.3) incorporates the pointwise and global approximation theorems of Bernstein operators, it is a very interesting estimate. Later in 1998, an inverse theorem of approximation for Bernstein operators in the following form was present in [3].

$$|B_nf(x)-f(x)|=\mathcal{O}\big((n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^{\alpha}\big)\Longleftrightarrow\omega^2_{\varphi^\lambda}(f,t)=\mathcal{O}(t^{\alpha}). \tag{1.4}$$

In this paper, we will establish the same result as (1.4) for the operators $C_n(f,s_n,x)$ defined in (1.1), but it must be restricted the sequence $\{s_n\}$ to be bounded.

2 Preliminary

We start with notation. Let $\delta_n^2(x)=\max\{\varphi^2(x),1/n\}$, $\varphi^2(x)=x(1-x)$, $\|f\|=\sup_{x\in I}|f(x)|$, and denoting

$$\begin{aligned}&\overrightarrow{\Delta}_h^2f(x)=f(x+2h)-2f(x+h)+f(x),\\&\omega^2_{\varphi^\lambda}(f,t)=\sup_{0<h\le t}\|\Delta^2_{h\varphi^\lambda}f(x)\|,\end{aligned}$$