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Necessary and Sufficient Conditions of Doubly Weighted Hardy-Littlewood-Sobolev Inequality

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Abstract. Using product and convolution theorems on Lorentz spaces, we characterize the sufficient and necessary conditions which ensure the validity of the doubly weighted Hardy-Littlewood-Sobolev inequality. It should be pointed out that we consider whole ranges of p and q, i.e., $0<p\le\infty$ and $0<q\le\infty$.

Key Words: Hölder's inequality, Young's inequality, Hardy-Littlewood-Sobolev inequality, Lorentz space.

AMS Subject Classifications: 42B20, 42B35

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1 Introduction

The Riesz potential operator

$$I_\alpha(f)(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy,$$

also called fractional integral operator, is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, provided that $1<p<q<\infty$ and $0<\alpha<n$ satisfy

$$\frac{1}{p}-\frac{\alpha}{n}=\frac{1}{q}.$$

According to the property of L^p space, we have $(L^p)^*=L^{p'}$, $1\le p<\infty$. Thus the $L^p\to L^q$-boundedness of I_s is equivalent to Theorem 1.1.

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Theorem 1.1. *Suppose that* $f\in L^{p_1}(\mathbb{R}^n)$ *and* $g\in L^{p_2}(\mathbb{R}^n)$. *If* $1<p_1,p_2<\infty$ *and* $0<\alpha<n$ *satisfy*

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{\alpha}{n}=2,$$

then

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x_1)g(x_2)}{|x_1-x_2|^\alpha}dx_1dx_2\right|\le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)} \tag{1.1}$$

holds.

Theorem 1.1 was obtained by Hardy and Littlewood for the case $n=1$ in [2] and by Sobolev for general n in [6]. Therefore, the inequality (1.1) is usually called the Hardy-Littlewood-Sobolev inequality in the literature. Stein and Weiss [7] considered the doubly weighted Hardy-Littlewood-Sobolev inequality and obtained Theorem 1.2 as follows.

Theorem 1.2. *If* $1<p,q<\infty$ *and* α, β, γ *satisfy the following conditions,*

$$\frac{1}{p}+\frac{1}{q}+\frac{\alpha+\beta+\gamma}{n}=2, \tag{1.2a}$$

$$\alpha+\gamma\ge 0,\quad \alpha<\frac{n}{p'},\quad \gamma<\frac{n}{q'},\quad \beta<n, \tag{1.2b}$$

$$\frac{1}{p}+\frac{1}{q}\ge 1, \tag{1.2c}$$

then

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\beta|y|^\gamma}dxdy\right|\le C\|f\|_{L^{p}(\mathbb{R}^n)}\|g\|_{L^{q}(\mathbb{R}^n)} \tag{1.3}$$

holds for $f\in L^p(\mathbb{R}^n)$ *and* $g\in L^q(\mathbb{R}^n)$.

Remark 1.1. In fact, both conditions (1.2a) and (1.2b) can easily imply

$$0<\beta<n. \tag{1.4}$$

It was the reason why Stein regarded (1.4) as one of the conditions in [7].

Clearly, the conditions (1.2a), (1.2b) and (1.2c) are sufficient conditions which ensure the validity of the doubly weighted Hardy-Littlewood-Sobolev inequality. In this paper, we use novel methods and ideas to investigate the sufficient and necessary conditions which makes the doubly weighted Hardy-Littlewood-Sobolev inequality hold. In particular, we apply the properties of product and convolution of two functions on Lorentz spaces to establish the doubly weighted Hardy-Littlewood-Sobolev inequality. It should be pointed out that we will consider whole ranges of p and q, i.e., $0<p\le\infty$ and $0<q\le\infty$, which cover Theorem 1.1 and Theorem 1.2, and provide us with new insightful information. The Hardy-Littlewood-Sobolev inequality is widely used in Harmonic Analysis, as well as in partial differential equation.

Now we formulate our main results as follows: