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Residues of Logarithmic Differential Forms in Complex Analysis and Geometry

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Abstract. In the article, we discuss basic concepts of the residue theory of logarithmic and multi-logarithmic differential forms, and describe some aspects of the theory, developed by the author in the past few years. In particular, we introduce the notion of logarithmic differential forms with the use of the classical de Rham lemma and give an explicit description of regular meromorphic differential forms in terms of residues of logarithmic or multi-logarithmic differential forms with respect to hypersurfaces, complete intersections or pure-dimensional Cohen-Macaulay spaces. Among other things, several useful applications are considered, which are related with the theory of holonomic $\mathscr{D}$-modules, the theory of Hodge structures, the theory of residual currents and others.

Key Words: Logarithmic differential forms, de Rham complex, regular meromorphic forms, holonomic $\mathscr{D}$-modules, Poincaré lemma, mixed Hodge structure, residual currents.

AMS Subject Classifications: 32S25, 14F10, 14F40, 58K45, 58K70

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1 Introduction

The purpose of the present notes is to sketch broad outlines of basic concepts of the residue theory of logarithmic and multi-logarithmic differential forms, and to describe some of little-known aspects of this theory, developed by the author in the past few years. In particular, we introduce the notion of logarithmic differential forms with the use of the classical de Rham lemma and then briefly review the theory of residue originated by H. Poincaré, J. Leray, K. Saito and others in various settings. Then we construct the sheaves of multi-logarithmic differential forms with respect to arbitrary reduced pure-dimensional Cohen-Macaulay space and describe their residues. In a certain sense, the set of all such forms could be viewed as the *universal* domain of definition of the residue

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map in the framework of the general residue theory. Among other things, we give an explicit description of regular meromorphic differential forms in terms of residues of logarithmic or multi-logarithmic differential forms with respect to hypersurfaces, complete intersections or pure-dimensional Cohen-Macaulay spaces.

It seems reasonable to say that further generalizations and interpretations of the notion of logarithmic differential forms and their residues have a large number of interesting applications in different contexts and settings, among which it should be mentioned the theory of arrangements of hyperplanes and hypersurfaces, the theory of index of vector fields and differential forms, deformation theory, tropical geometry, the theory of resolutions of singularities, the theory of integral representations and residual currents, etc. Moreover, in a certain sense the theory of b-functions can be considered as a noncommutative analog or an extension of the theory of logarithmic differential forms to the category of $\mathscr{D}$-modules, etc.

The paper is organized as follows. In the first sections we discuss simple properties of logarithmic differential forms with poles along a reduced divisor. In particular, we give an unorthodox definition of this notion with the use of a version of the classical de Rham lemma adopted to the case of singular hypersurfaces. Then we consider some applications involving a logarithmic version of the classical Poincaré lemma and the classification problem of integrable holonomic $\mathscr{D}$-modules of Fuchsian and logarithmic type, etc. The basic concept of regular meromorphic differential forms is discussed in Section 4. In the next three sections we describe an explicit construction of residues for meromorphic forms with logarithmic poles along reduced divisors and for multilogarithmic forms with respect to complete intersections. As an application, in Section 8, we show how to describe the weight filtration on the logarithmic de Rham complex directly, without the use of Hironaka's resolution of singularities. In the final section quite a general construction of multi-logarithmic differential forms with respect to reduced puredimensional Cohen-Macaulay spaces is presented and some relations with the theory of residue currents are discussed.

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2 The de Rham lemma

Let M be a complex manifold of dimension m, $m \geq 1$, and let $X \subset M$ be an subset in an open neighborhood U of the distinguished point $\mathfrak{o} \in U \subset M$ defined by a sequence of functions $f_1,\cdots,f_k \in \mathcal{O}_U$. We denote by Ω^p_X, $p \geq 0$, the sheaves of germs of *regular holomorphic* differential p-forms on X; they are defined as the restriction to X of the quotient module

$$\Omega^p_X = \Omega^p_U / ((f_1,\cdots,f_k)\Omega^p_U + df_1 \wedge \Omega^{p-1}_U + \cdots + df_k \wedge \Omega^{p-1}_U)\big|_X.$$