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# A Sufficient Condition for Rigidity in Extremality of Teichmüller Equivalence Classes by Schwarzian Derivative

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**Abstract.** The Strebel point is a Teichmüller equivalence class in the Teichmüller space that has a certain rigidity in the extremality of the maximal dilatation. In this paper, we give a sufficient condition in terms of the Schwarzian derivative for a Teichmüller equivalence class of the universal Teichmüller space under which the class is a Strebel point. As an application, we construct a Teichmüller equivalence class that is a Strebel point and that is not an asymptotically conformal class.

**Key Words**: Strebel points, the Schwarzian derivative, asymptotically conformal maps.

**AMS Subject Classifications**: 30F60, 30C62

## 1 Introduction

The Teichmüller space is the deformation space of marked Riemann surfaces. Indeed, the space is defined as the quotient space of the family of marked Riemann surfaces by a certain topological equivalence relation. Each element of the Teichmüller space is called the Teichmüller equivalence class. Especially, the Teichmüller space of the hyperbolic plane is called the universal Teichmüller space, denoted by $T$.

It is known that each Teichmüller equivalence class has a quasiconformal mapping with smallest maximal dilatation in its class, which is called extremal. It is generally difficult to find an extremal quasiconformal mapping in each Teichmüller equivalence class. However the extremal quasiconformal mapping is uniquely determined in a certain Teichmüller equivalence class named the Strebel point, which means the class where the boundary dilatation is less than the maximal dilatation. This result is called Strebel's frame mapping theorem (see Chapter 4 in [5]). There exist several studies of the Strebel

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point. For example, Lakic [6] proved that the set of Strebel points is open and dense in the Teichmüller space and Earle and Li [4] showed that a Teichmüller equivalence class $\tau$ is a Strebel point if and only if there exists exactly one geodesic connecting the basepoint to $\tau$ in the Teichmüller distance. Hence the Strebel point is an important concept for the Teichmüller theory.

In this paper, we deal with Strebel points of the universal Teichmüller space $T$. In other words, we will give a sufficient condition for a Teichmüller equivalence class under which the class is a Strebel point. We take $\Delta^* = \{|z| > 1\} \cup \{\infty\}$ as the model of the hyperbolic plane. The main tool is the Schwarzian derivative, which induces a homeomorphic embedding of $T$ into the Banach space $\mathcal{B}$ of holomorphic functions on the unit disk $\Delta = \{|z| < 1\}$ with finite hyperbolic sup-norm

$$\|\varphi\|_{\mathcal{B}} = \sup_{z \in \Delta} (1-|z|^2)^2 |\varphi(z)|.$$

We have preliminaries in Section 2 and Section 3. Section 2 is devoted to introduce some properties of the Schwarzian derivative in the Teichmüller theory. Especially, we discuss the quasiconformal extensibility of a meromorphic function on $\Delta$ to the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by estimating the hyperbolic sup-norm of its Schwarzian derivative. Section 3 contains a brief summary of the universal Teichmüller space and Strebel points. We introduce an example of Strebel points named the asymptotically conformal class, whose boundary dilatation vanishes. The set $T_0$ of asymptotically conformal classes is a closed submanifold of $T$. In fact, $T_0$ is embedded into a closed subspace $\mathcal{B}_0$ of the Banach space $\mathcal{B}$, where each element of $\mathcal{B}$ vanishes in the semi-norm

$$\|\varphi\|_{\mathcal{B}_0} = \limsup_{|z| \to 1} (1-|z|^2)^2 |\varphi(z)|.$$

We call this norm the boundary hyperbolic sup-norm. The subspace $T_0$ is studied in its analytic and metric structure (see [1, 2]).

In Section 4 our main result is stated and proved. We will require a comparison of the hyperbolic and boundary hyperbolic sup-norm in the sufficient condition. As an application, we construct a Teichmüller equivalence class that is a Strebel point and that is not an asymptotically conformal class. Since it is generally difficult to find such a class specifically, it is considered that such an example is significant.

## 2 Schwarzian derivative and quasiconformal mappings

In this section, we discuss a relation between the Schwarzian derivative and quasiconformal mappings. Let $f$ be a meromorphic function in a simply connected domain $A$ of $\hat{\mathbb{C}}$. Then the expression

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$