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On Potentially Graphical Sequences of $G-E(H)$

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Abstract. A loopless graph on n vertices in which vertices are connected at least by a and at most by b edges is called a (a,b,n)-*graph*. A (b,b,n)-*graph* is called (b,n)-*graph* and is denoted by K_n^b (it is a complete graph), its complement by $\overline{K}_n^b$. A non increasing sequence $\pi=(d_1,\cdots,d_n)$ of nonnegative integers is said to be (a,b,n) graphic if it is realizable by an (a,b,n)-graph. We say a simple graphic sequence $\pi=(d_1,\cdots,d_n)$ is potentially $K_4-K_2\cup K_2$-graphic if it has a a realization containing an $K_4-K_2\cup K_2$ as a subgraph where K_4 is a complete graph on four vertices and $K_2\cup K_2$ is a set of independent edges. In this paper, we find the smallest degree sum such that every n-term graphical sequence contains $K_4-K_2\cup K_2$ as subgraph.

Key Words: Graph, (a,b,n)-*graph*, potentially graphical sequences.

AMS Subject Classifications: 05C07

1 Introduction

Let $G(V,E)$ be a simple graph (a graph without multiple edges and loops) with n vertices and m edges having vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$. The set of all non-increasing non-negative integer sequences $\pi=(d_1,d_2,\cdots,d_n)$ is denoted by NS_n. A sequence $\pi\in NS_n$ is said to be graphic if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π. The set of all graphic sequences in NS_n is denoted by GS_n. There are several famous results, Havel and Hakimi [7,8] and Erdös and Gallai [2] which give necessary and sufficient conditions for a sequence $\pi=(d_1,d_2,\cdots,d_n)$ to be the degree sequence of a simple graph G. Another characterization of graphical sequences can be seen in Pirzada and Yin Jian Hu [15]. A graphical sequence π is potentially H-graphical if there is a realization of π containing H as a subgraph, while π is forcibly H graphical if every realization of π contains H as a subgraph. A sequence $\pi=(d_1,d_2,\cdots,d_n)$

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is said to be potentially K_{r+1}- graphic if there is a realization G of π containing K_{r+1} as a subgraph. It is shown in [4] that if π is a graphic sequence with a realization G containing H as a subgraph, then there is a realization G of π containing H with the vertices of H having $|V(H)|$ largest degree of π. If π has a realization in which the $r+1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1}-graphic. We know that a graphic sequence π is potentially K_{k+1}-graphic if and only if π is potentially A_{k+1}-graphic [17]. The disjoint union of the graphs G_1 and G_2 is defined by $G_1 \bigcup G_2$. Let K_k and C_k respectively denote a complete graph on k vertices and a cycle on k vertices.

In order to prove our main results, the following notations, definitions and results are needed. Let $G=(V(G),E(G))$ be a simple graph with vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$. The degree of v_i is denoted by d_i for $1\leq i\leq n$. Then $\pi=(d_1,d_2,\cdots,d_n)$ is the degree sequence of G, where $d_1,d_2,\cdots,d_n$ may be not in increasing order. In order to prove our main results, we also need the following notations and results. Let $\pi=(d_1,d_2,\cdots,d_n)\in NS_n$, $1\leq k\leq n$. Let

$$\begin{aligned} \pi'' &= (d_1-1,\cdots,d_{k-1}-1,\cdots,d_{d_k+1}-1,d_{d_k+2},\cdots,d_n), && \text{if } d_k\geq k, \\ &= (d_1-1,\cdots,d_k-1,\cdots,d_{d_k+1},\cdots,d_{k-1},d_{k+1},d_n), && \text{if } d_k<k. \end{aligned}$$

Denote $\pi'_k=(d_1^{i'}, d_2^{i'},\cdots,d_{n-1}^{i'})$, $1\leq i'\leq n$, where $d_1^{i'},d_2^{i'},\cdots,d_{n-1}^{i'}$ is a rearrangement of the $n-1$ terms of π''. Then π'' is called the residual sequence obtained by laying off d_k from π.

Definition 1.1. A wheel graph W_n is a graph with n vertices $(n\geq 4)$ formed by connecting a single vertex to all vertices of an $(n-1)$ cycle. A wheel graph on 4 and 5 vertices are shown in Fig. 1 below.

In 1960 Erdős and Gallai gave the following necessary and sufficient condition.

Theorem 1.1 (see Erdős, Gallai [2]). *Let $n\geq 1$. An even sequence $\pi=(d_1,\cdots,d_n)$ is graphical if and only if*

$$\sum_{i=1}^{k} d_i \leq k(k-1)+\sum_{i=k+1}^{n}\min(d_i,k)$$

is satisfied for each integer k, $1\leq k\leq n$.

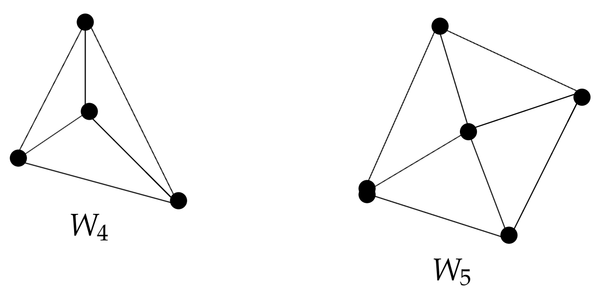

Figure 1:

Theorem 1.2 (see [4]). *If $\pi=(d_1,d_2,\cdots,d_n)$ is the graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π.*

2 (a,b,n)-graphic sequences

In this section a,b,n,m,n and l denote non-negative integers with $b\geq a$. An *(a,b,n)-graph denoted by G^** is a loopless graph in which different vertices are connected at least by a and at most by b edges [9, 10]. A *(b,b,m)-graph* on m vertices denoted by K_m^b and is called shortly *b-graph*. Clearly, $K_m^1=K_m$, where K_m is the *complete graph* on m vertices. A nonincreasing sequence $\pi=(d_1,\cdots,d_n)$ of nonnegative integers is said to be *b-graphic* if it is the degree sequence of an b-graph G^* on n vertices and such a graph G^* is referred to as a *realization* of π.

The following three results due to Chungphaisan [1] are generalizations from 1-graphs to b-graphs of three well-known results, one by Erdös and Gallai [2], one by Kleitman and Wang [11], one by Fulkerson, Hoffman and McAndrew [5].

Theorem 2.1 (see [1]). *Let $\pi=(d_1,\cdots,d_n)$ be a nonincreasing sequence of non-negative integers, where the sum of the elements of π is even. Then π is b-graphic if and only if for each positive integer $t\leq n$,*

$$\sum_{i=1}^{t}d_i\leq bt(t-1)+\sum_{i=t+1}^{n}\min(bt,d_i).$$

Let $\pi=(d_1,\cdots,d_n)$ be a nonincreasing sequence of nonnegative integers with $d_1\leq\sum_{i=2}^{n}\min(b,d_i)$. Define $\pi_k^{'}=(d_1',\cdots,d_{n-1}')$ to be the nonincreasing rearrangement of the sequence obtained from $(d_1,\cdots,d_{k-1},d_{k+1},\cdots,d_n)$ reducing by 1 the remaining largest term that has not already been reduced b times, and repeating the procedure d_k times. $\pi_k^{'}$ is called the residual sequence obtained from π by laying off d_k.

Theorem 2.2 (see [11]). *π is b-graphic if and only if π_k' is b-graphic.*

Theorem 2.3 (see [1]). *Let π be an b-graphic sequence, and let G and G' be realizations of π. Then there is a sequence of r-exchanges, $E_1,\cdots,E_k$ such that the application of these b-exchanges to G in order will result in G'.*

An extremal problem for 1-graphic sequences to be potentially K_l^1-graphic was considered by Erdős, Jacobson and Lehel [3], and solved by Gould et al. [6] and Li et al. [13, 14]. Recently, Yin [18] generalized this extremal problem and the Erdős-Jacobson-Lehel conjecture from 1-graphs to b-graphs.

An (a,b,n)-graphic sequence π is said to be potentially K_m^b-graphic if there exists a realization G of π containing K_m^b as a subgraph. If there exists a realization G of π containing K_m^b on the m vertices of highest degree in G, then π is said to be potentially A_m^b-graphic. As a special case of Lemma 2.1 in [18], Yin showed that a b-graphic sequence is potentially K_m^b-graphic if and only if it is potentially A_m^b-graphic. If $\pi=(d_1,\cdots,d_n)$ has a realization G containing $S_{l,m}^b$ on those vertices having degree $d_1,\cdots,d_{l+m}$ such that the vertices of K_l^b have degree $d_1,\cdots,d_l$ and the vertices of $\overline{K}_m^b$ have degree $d_{l+1},\cdots,d_{l+m}$, then π is potentially $A_{l,m}^b$-graphic.

Theorem 2.4 (see Yin [19]). *Let $n\geq r+s$ and let $\pi=(d_1,\cdots,d_n)$ be a nonincreasing graphic sequence. If $d_{r+s}\geq r+s-2$, then π is potentially $A_{r,s}$-graphic.*

In the same paper Yin published a Havel-Hakimi type algorithm constructing the corresponding $S_{r,s}$-graph.

In 2014 Pirzada and Chat proved the following assertion:

Theorem 2.5 (Pirzada, Chat [16]). *If G_1 is a realization of $\pi_1=(d_1^1,\cdots,d_m^1)$, containing K_p as a subgraph and G_2 is a realization of $\pi_2=(d_1^2,\cdots,d_n^2)$ containing K_q as a subgraph, then the degree sequence $\pi=(d_1,\cdots,d_{m+n})$ of the join of G_1 and G_2 is K_{p+q}-graphic.*

Problem 2.1. Let H be the graph and n be the positive integer. Determine the smallest even integer $\sigma(H,n)$ such that every n-term graphic sequence contains H as a subgraph.

The purpose of this paper is to solve problem 6 by taking $H=W_4-(K_2\cup K_2)$ and we also obtain the graphic sequence of the graph when only edge size of the graph and degree of first vertex of the non-increasing sequence of integers is given. Next we find the Necessary and sufficient condition for a b-graphic sequence to be b-star graphic.

Remark 2.1. Let cl_{2m} be the circular ladder graph with $2m$ vertices, $m\geq 3$ formed by taking two copies of the cycle C_m with corresponding vertices from each copy of C_m being adjacent. Let $(K_{2m}-cl_{2m})$ be the graph obtained from K_{2m} by removing the edges of cl_{2m}. For $m\geq 3$, it can easily be seen that $(K_{2m}-cl_{2m})$ is a $2m-4$ regular graph on $2m$ vertices and the number of edges in this graph be $2m(m-2)$. Fig. 2 below is an example of ladder graph on 8 vertices.

Remark 2.2. Let K_n^b be the complete b-graph, then the total number of edges in the graph be $\frac{1}{2}bn(n-1)$. For example if we take K_4^3 the complete 3-graph, then the total number of edges in this graph be 18 as shown in figure below.

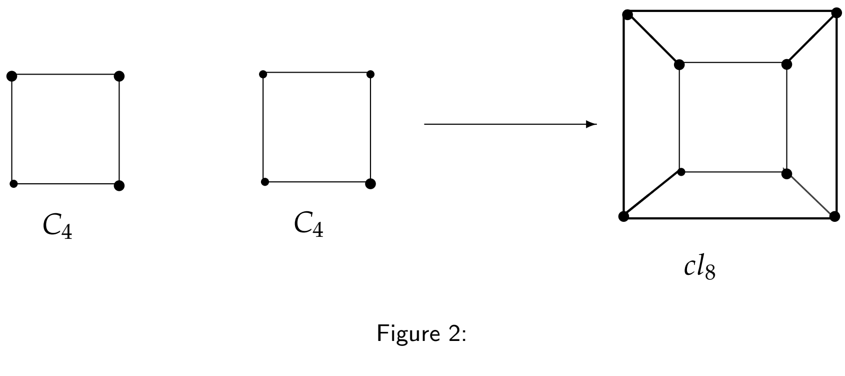

Figure 2:

3 Main results

We start with the following main results. In the following result, we obtain the graphic sequence and the degree sum of the graph when edge size and degree of first vertex is given.

Theorem 3.1. *If $\pi_1=(d_1,\cdots,d_n)$ be a graphic sequence satisfying $d_1=2(n-2)$ and $|E|^2=\frac{n}{2}\sum\limits_{i=1}^{2n}d_i^2$, $n\geq 3$, then, $\pi=(2(n-2))^{2n}$, $\sigma(\pi)=4n(n-2)$ and π realizes $K_{2n}-cl_{2n}$.*

Proof. Suppose $\pi=(d_1,d_2,\cdots,d_n)$ be a graphic sequence with $d_1=2(n-2)$, then there exists a graph G which realizes π. Now we have

$$\begin{aligned}
|E|^2=&\frac{n}{2}\sum_{i=1}^{2n}d_i^2\cdots\\
\Rightarrow&\frac{2}{n}|E|^2-\sum_{i=1}^{2n}d_i^2=0\\
\Rightarrow&d_1^2+d_2^2+\cdots+d_n^2+d_{n+1}^2+\cdots+d_{2n}^2-\frac{2}{n}|E|^2=0\\
\Rightarrow&d_1^2+d_2^2+\cdots+d_n^2+d_{n+1}^2+\cdots+d_{2n}^2-\frac{1}{2n}|2E|^2=0\\
\Rightarrow&d_1^2+d_2^2+\cdots+d_n^2+d_{n+1}^2+\cdots+d_{2n}^2-\frac{1}{2n}(d_1+d_2+\cdots+d_{2n})^2=0\\
\Rightarrow&d_1^2+d_2^2+\cdots+d_n^2+d_{n+1}^2+\cdots+d_{2n}^2-\frac{1}{2n}\big(d_1^2+d_2^2+\cdots+d_n^2+d_{n+1}^2\\
&\quad+\cdots+d_{2n}^2+2(d_1d_2+d_1d_3+\cdots+d_1d_{2n})+2(d_2d_3+d_2d_4\\
&\quad+\cdots+d_2d_{2n})+\cdots+2(d_{2n-2}d_{2n-1}+d_{2n-2}d_{2n})+2d_{2n-1}d_{2n}\big)=0\\
\Rightarrow&\frac{1}{2n}(2n-1)d_1^2+\frac{1}{2n}(2n-1)d_2^2+\cdots+\frac{1}{2n}(2n-1)d_n^2+\frac{1}{2n}(2n-1)d_{n+1}^2\\
&\quad+\cdots+\frac{1}{2n}(2n-1)d_{2n}^2-\frac{1}{2n}\big(d_1^2+d_2^2+\cdots+d_n^2+d_{n+1}^2+\cdots+d_{2n}^2\big)\\
&\quad+2(d_1d_2+d_2d_3+\cdots+d_1d_{2n})+2(d_2d_3+d_2d_4\\
&\quad+\cdots+d_2d_{2n})+\cdots+2(d_{2n-2}d_{2n-1}+d_{2n-2}d_{2n})+2d_{2n-1}d_{2n}\big)=0\\
\Rightarrow&\frac{1}{2n}\big((d_1-d_2)^2+(d_1-d_3)^2+\cdots\\
&\quad+(d_1-d_n)^2+(d_1-d_{n+1})^2+\cdots+(d_1-d_{2n})^2\big)=0.
\end{aligned}$$

From this equation we see that every term of L. H. S is positive for every i, j and R. H. S is equal to zero. Therefore above equation is possible when $d_i=d_j$ for every $i,j=1,2,\cdots,2n$. Since $d_1=2(n-2)$, therefore $d_2=d_1=2(n-2)$ and hence $d_k=2(n-2)$ for all $k=1,2,\cdots,2n$. Thus $\pi=(2(n-2))^{2n}$. Further $\sigma(\pi)=2n\times 2(n-2)=4n(n-2)$.

Figure 3:

Now from Remark 2.1, we see that π is the graphic sequence of the graph $K_{2n} - cl_{2n}$. Therefore π realizes $K_{2n} - cl_{2n}$. □

In the following result, we find the smallest graphic sum such that every n-term graphic sequence contains $W_4 - (K_2 \cup K_2)$ as a subgraph.

Theorem 3.2. *If π be the graphic sequence with $\sigma(\pi) \geq 3n-1$ if n is odd and $\sigma(\pi) \geq 3n-2$ if n is even, then π is potentially $W_4 - (K_2 \cup K_2)$-graphic.*

Proof. Let π be the graphic sequence. then there exists a graph G which realizes π. We have to show that if $\sigma(\pi) \geq 3n-1$ and $\sigma(\pi) \geq 3n-2$, then every n-term graphic sequence contains $W_4 - (K_2 \cup K_2)$ as a subgraph, where $K_2 \cup K_2$ is the matching in G. To prove the result we use induction on n and we start induction for $n \geq 4$. For $n = 4$, then by the assumption we have $|E| \geq 5$, therefore in this case the realization G of π contains $W_4 - (K_2 \cup K_2)$ as a subgraph as illustrated in Fig. 3.

Clearly from Fig. 3, there are exactly two graphs with $|G| = 4$ and $|E| \geq 5$ and both these graphs contains $W_4 - (K_2 \cup K_2)$ as a subgraph. Thus π is potentially $W_4 - (K_2 \cup K_2)$-graphic. Now for $n = 5$, therefore from give assumption we have $|E| \geq 7$, then there are exactly four graphic sequences $(4,3^2,2^2)$,$(4,3^3,1)$,$(3^4,2)$ and $(4^2,2^3)$ with $\sigma(\pi) = 14$ and each of these graphic sequences have a realization G containing $W_4 - (K_2 \cup K_2)$ as a subgraph as illustrated in Fig. 4.

All these graphs contains $W_4 - (K_2 \cup K_2)$ and the result is true in this case also. Now assume that the result is true for all graphic sequences of n-terms and we now consider the graphic sequences of $n+1$ terms. Now if the graphic sequence π contains a vertex of degree equal to 1, then remove it and adjust the new sequence π'. By induction realization G of π' must contain a $W_4 - (K_2 \cup K_2)$. We know that for $n \geq 6$ smallest degree sum such that every n-term graphic sequence contains a clique on three vertices is $2n$. Since $\sigma(\pi) = 16$ for $n = 6$ which is greater than the smallest degree sum such that every 6-term graphic sequence contains a clique on 3 vertices which can be obtained in a realization using the two vertices of highest degree. Let this complete graph graph on three vertices has vertices y_1,y_2 and y_3 and assume that these two vertices of highest degree in the graph are y_1 and y_2. Thus y_1 and y_2 have at least one more adjacency in the graph say y_1 is adjacent to x and y_2 is adjacent to y as shown in Fig. 5.

Now we consider the following cases:

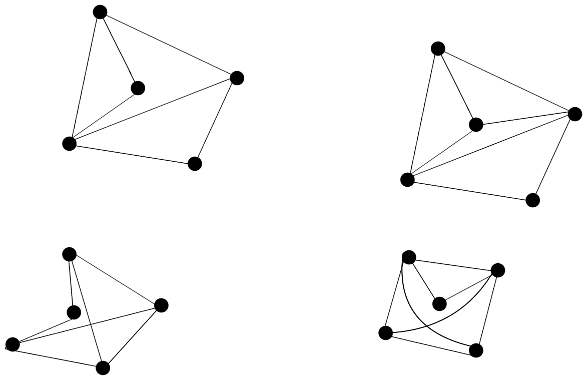

Figure 4:

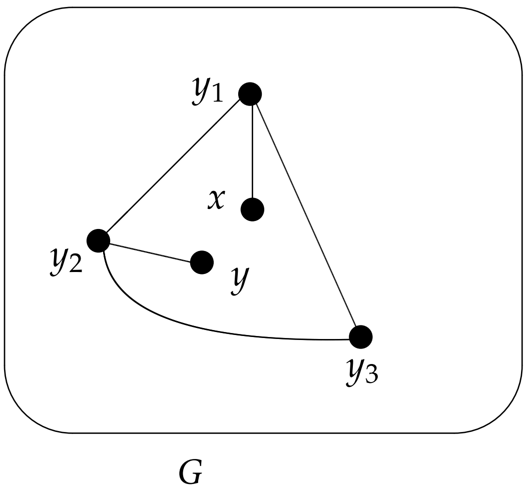

Figure 5:

Case I. If $x=y$, then G contains the subgraph $W_4-(K_2\cup K_2)$, therefore the result is true in this case.

Case II. If $x\neq y$, we consider two subcases.

Subcase 1. Suppose x and y have common vertex w such that xw and $yw\in E(G)$. Then we see that wy_1 and xy_3 are not in realization G of π, since otherwise we get a realization G containing $W_4-(K_2\cup K_2)$. Then by EDT by removing the independent edges $K_2\cup K_2$ (xw and y_1y_3) and inserts the independent edges $K_2\cup K_2$ (wy_1 and xy_3) produces a realization G' of π containing a $W_4-(K_2\cup K_2)$ on the vertex set $S=\{y_1,y_2,w,y\}$.

Subcase 2. Suppose that x and y have no common adjacency of a clique on three vertices. Suppose x is adjacent to x' and y is adjacent to y' such that $x'\neq y'$. Now suppose that $x'y'\notin$

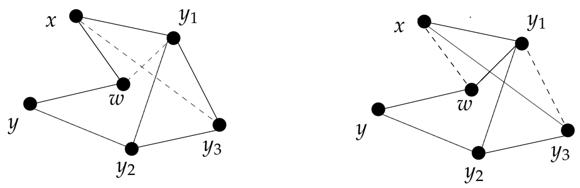

Figure 6:

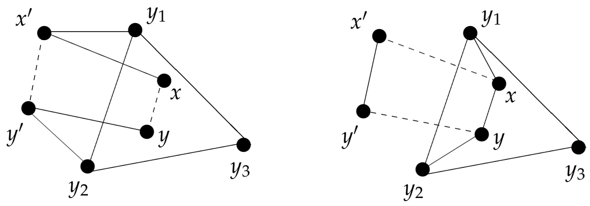

Figure 7:

$E(G)$, then by EDT that removes the independent edges $K_2 \cup K_2$ (xx' and yy') and inserts the independent edges $x'y'$ and xy produces a realization G of π containing $W_4-(K_2\cup K_2)$. These two subcases are illustrated in Figs. 6 and 7 below.

Subcase 3. Now if $x'y' \in E(G)$, then again it is easy to see that the independent edges x_1y_1 and $xy_3 \notin E(G)$, since otherwise $W_4-(K_2\cup K_2)$ would exist. Therefore again by EDT that removes the independent edges y_1y_3 and xx' and inserts the independent edges y_1x' and xy_2 produces a realization G of π containing $W_4-(K_2\cup K_2)$. Thus in all cases $W_4-(K_2\cup K_2)$ was produced in some realization of π and therefore the graphic sequence π is potentially $W_4-(K_2\cup K_2)$ and hence the result is proved. □

Example 3.1. Let $\pi_1=(4,3^3,1)$ be the non-negative sequence. Then clearly it is graphic with $\sigma(\pi)=14$. Therefore by above theorem realization of π contains $H=W_4-(K_2\cup K_2)$ as a subgraph. Thus π_1 is potentially H-graphic.

Example 3.2. Let $\pi_2=(3^2,2^3)$ be the non-negative sequence. Then clearly it is graphic with $\sigma(\pi)=12$. Therefore by above theorem every 5-term graphic sequence of π does not contains $H=W_4-(K_2\cup K_2)$ as a subgraph. Thus every 5-term graphic sequence of π_2 does not contain H as a subgraph.

Next we obtain the following results on (a,b,n)-graphic sequences.

Lemma 3.1. *An (a,b,n)-graph G^* is r-regular if and only if $\frac{4}{n}|E|^2=d_1^2+d_2^2+\cdots+d_n^2$.*

Proof. Suppose (a,b,n)-graph G^* is r-regular, then $2|E|=nr$ and $d_i=r$ for all $i=1,2,\cdots,n$. Therefore, we have

$$d_1^2=r^2, \quad d_2^2=r^2,\cdots,d_n^2=r^2.$$

Therefore, $d_1^2+d_2^2+\cdots+d_n^2=nr^2$.

Also we have $|E|^2=\frac{n^2r^2}{4}=\frac{n}{4}nr^2\Rightarrow\frac{4}{n}|E|^2=d_1^2+d_2^2+\cdots+d_n^2$. In similar lines we can prove the converse part of this lemma. □

Theorem 3.3. *Let G^* be (a,b,n)-graph with $n\geq 2$, then G^* is a b-complete graph K_n^b iff*

$$\sum_{i=1}^{n} d_i^2=|E|\left(b(n-2)+\frac{2|E|}{n-1}\right).$$

Proof. Suppose that G^* is b-complete on n vertices with $n\geq 2$. Then by Remark 2.2, $|E|=\frac{1}{2}nb(n-1)$. Therefore

$$\begin{aligned}
&\Rightarrow 2|E|(n-2)=nb(n-1)(n-2)\\
&\Rightarrow 2|E|(n-2)+2|E|n=nb(n-1)(n-2)+2|E|n\\
&\Rightarrow 4|E|n-4|E|=n\big(b(n-1)(n-2)+2|E|\big)\\
&\Rightarrow \frac{4|E|^2}{n}=|E|\left(b(n-2)+\frac{2|E|}{n-1}\right).
\end{aligned}$$

Conversely suppose that the graph $G(a,b,n)$ holds

$$\sum_{i=1}^{n} d_i^2=|E|\left(b(n-2)+\frac{2|E|}{n-1}\right).$$

Therefore by Lemma 3.1, we have

$$\begin{aligned}
\frac{4|E|^2}{n}&=|E|\left(b(n-2)+\frac{2|E|}{n-1}\right)\\
&\Rightarrow|E|=\frac{bn(n-2)}{4}+\frac{n|E|}{2(n-1)}\\
&\Rightarrow|E|\big(1-\frac{n}{2(n-1)}\big)=\frac{n}{4}b(n-2)\\
&\Rightarrow|E|\big(\frac{2n-2-n}{2(n-1)}\big)=\frac{n}{4}b(n-2)\\
&\Rightarrow|E|\big(\frac{1}{(n-1)}\big)=\frac{n}{2}b\\
&\Rightarrow|E|=\frac{nb(n-1)}{2}.
\end{aligned}$$

Therefore, by Remark 2.2 G^* is b-complete graph K_n^b. □

Theorem 3.4. *If a multigraph G^* is b-complete bipartite graph $K_{r,s}^b$, then G^* is a b-star graph $K_{1,n-1}^b$ if*

$$\sum_{i=1}^{n} d_i^2=|E|\left(b(n-2)+\frac{2|E|}{n-1}\right).$$

Proof. Suppose that the multigraph $G^*=K_{r,s}^{(b)}$ where s is greater than 1. Since there are r-vertices each of whose degree is $b\times s$ and s vertices each of whose degree is $b\times r$. Therefore we have $\sum_{i=1}^{n} d_i^2 = r(bs)^2+s(br)^2 = rb^2s^2+sb^2r^2 = b^2(rs^2+sr^2) = b^2rs(r+s)$. Now, from the above it follows that $n=r+s$ and $|E|=brs$. Therefore, we have

$$\begin{aligned}|E|\Big(b(n-2)+\frac{2|E|}{n-1}\Big) &= brs\Big(b(r+s-2)+\frac{2brs}{n-1}\Big)\\ &= brs\Big(\frac{b(r+s-2)(r+s-1)+2brs}{r+s-1}\Big)\\ &= brs\Big(\frac{b(r^2+s^2+2rs-3r-3s+2)+2brs}{r+s-1}\Big)\\ &= b^2rs\Big(\frac{r^2+s^2+4rs-3r-3s+2}{r+s-1}\Big).\end{aligned}$$

Now suppose that

$$\begin{aligned}\sum_{i=1}^{n} d_i^2 &= |E|\Big(b(n-2)+\frac{2|E|}{n-1}\Big)\\ &= b^2rs\Big(\frac{r^2+s^2+4rs-3r-3s+2}{r+s-1}\Big)\\ &\Rightarrow b^2rs(r+s) = b^2rs\Big(\frac{r^2+s^2+4rs-3r-3s+2}{r+s-1}\Big)\\ &\Rightarrow r+s = \frac{r^2+s^2+4rs-3r-3s+2}{r+s-1}\\ &\Rightarrow (r+s)^2-(r+s) = r^2+s^2+4rs-3r\\ &\Rightarrow 2rs-2(r+s)+2=0\\ &\Rightarrow r(s-1)=1\\ &\Rightarrow r=1.\end{aligned}$$

Thus we have proved that a multigraph G^* is a b star graph if

$$\sum_{i=1}^{n} d_i^2 = |E|\Big(b(n-2)+\frac{2|E|}{n-1}\Big).$$

So, we complete the proof. □

Corollary 3.1. Let G^* be complete multibibartite graph, if G^* is $K_{1,n-1}^{b}$, then

$$|E| = \frac{n}{4}\Big(\frac{2b(n-2)}{n}+\frac{2|E|}{n-1}\Big).$$

Proof. Suppose that a complete multi-bipartite graph is $K^b_{1,n-1}$. Then we have

$$
\begin{aligned}
&|E| = b(n-1) \Rightarrow 2|E|(n-2) = 2b(n-1)(n-2)\\
&\quad \Rightarrow 2|E|(n-2) + 2|E|n = 2b(n-1)(n-2) + 2|E|n \Rightarrow 4|E|(n-1) = 2b(n-1)(n-2) + \frac{2|E|n}{n-1}\\
&\quad \Rightarrow |E| = \frac{n}{4}\Big(\frac{2b(n-2)}{n} + \frac{2|E|}{n-1}\Big).
\end{aligned}
$$

Thus, we complete the proof. □

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