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Certain Integral Transforms of Generalized k-Bessel Function

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Abstract. The objective of this note is to provide some (potentially useful) integral transforms (for example, Euler, Laplace, Whittaker etc.) associated with the generalized k-Bessel function defined by Saiful and Nisar [3]. We have also discussed some other transforms as special cases of our main results.

Key Words: Gamma function, k-Bessel function, generalized k-Bessel function, integral transforms.

AMS Subject Classifications: 33C10, 65R10, 26A33

1 Introduction

The Bessel function of fist kind has the power series representation of the form [4]:

$$J_v(z)=\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{2}\right)^{2k+v}}{\Gamma(k+v+1)k!}, \tag{1.1}$$

Romero et al. [16] introduced the k-Bessel function of the first kind defined by the series

$$J_{k,v}^{\gamma,\lambda}(x)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+v+1)}\frac{(-1)^n(x/2)^n}{(n!)^2}, \tag{1.2}$$

where $k\in\mathbb{R}$; $\alpha,\lambda,\gamma,v\in C$; $\Re(\lambda)>0$ and $\Re(v)>0$.

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Very recently, Saiful and Nisar [3] gave a new generalization of *k*-Bessel function called the generalized *k*-Bessel function of the first kind defined for $k\in\mathbb{R}$; $\sigma,\gamma,v,c,b\in\mathbb{C}$; $\Re(\sigma)>0$, $\Re(v)>0$ as:

$$J_{k,v}^{b,c,\gamma,\sigma}(z)=\sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\sigma n+v+\frac{b+1}{2}\right)}\frac{(z/2)^{v+2n}}{(n!)^2}, \tag{1.3}$$

where the *k*-Pochhammer symbol $(\gamma)_{n,k}$ is defined by [1]:

$$(\gamma)_{\nu,k}=\frac{\Gamma_k(\gamma+\nu k)}{\Gamma_k(\gamma)},\quad (\gamma\in\mathbb{C}\setminus\{0\}), \tag{1.4}$$

and the *k*-gamma function has the relation

$$\Gamma_k(z)=k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right), \tag{1.5}$$

such that $\Gamma_k(z)\to\Gamma(z)$ if $k\to 1$.

The generalized hypergeometric function represented as follows [6]:

$${}_pF_q\left[\begin{array}{c}(\alpha_p),\\(\beta_q),\end{array} z\right]=\sum_{n=0}^{\infty}\frac{\Pi_{j=1}^{p}(\alpha_j)_n}{\Pi_{j=1}^{q}(\beta_j)_n}\frac{z^n}{n!}, \tag{1.6}$$

provided $p\le q$, $p=q+1$ and $|z|<1$ and $(\alpha)_n$ is well known Pochhammer symbol (see [6]). The Fox-Wright generalization ${}_p\Psi_q(z)$ of hypergeometric function ${}_pF_q$ is given by (c.f. [7–9,15]):

$$\begin{aligned}{}_p\Psi_q\left[\begin{array}{c}(\alpha_1,A_1),\cdots,(\alpha_p,A_p),\\(\beta_1,B_1),\cdots,(\beta_q,B_q),\end{array} z\right]&={}_p\Psi_q\left((\alpha_j,A_j)_{1,p};(b_j,\beta_j)_{1,q};z\right)\\&=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1n)\cdots\Gamma(\alpha_p+A_pn)}{\Gamma(\beta_1+B_1n)\cdots\Gamma(\beta_q+B_qn)}\frac{z^n}{n!},\end{aligned} \tag{1.7}$$

where $A_j>0$ $(j=1,2,\cdots,p)$; $B_j>0$ $(j=1,2,\cdots,q)$ and

$$1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\ge 0$$

for suitably bounded value of $|z|$.

The generalized *k*-Wright function introduced in [10] as: For $k\in\mathbb{R}^+$; $z\in\mathbb{C}$, $\alpha_i,\beta_j\in\mathbb{R}(\alpha_i,\beta_j\neq 0;\, i=1,2,\cdots,\, p;j=1,2,\cdots,q)$ and $(a_i+\alpha_i n),(b_j+\beta_j n)\in\mathbb{C}\setminus k\mathbb{Z}^-$

$${}_p\Psi_q^k(z)={}_p\Psi_q^k\left[\left(\begin{array}{cc}(a_i,\alpha_i)_{1,p} & \\ (b_j,\beta_j)_{1,q} & |z\end{array}\right)\right]=\sum_{n=0}^{\infty}\frac{\prod\limits_{i=1}^{p}\Gamma_k(a_i+\alpha_i n)}{\prod\limits_{j=}^{q}\Gamma_k(b_j+\beta_j n)}\frac{z^n}{n!}. \tag{1.8}$$

Also, we recall here the following definitions:

Definition 1.1. Euler Transform: Let $\alpha,\beta\in\mathbb{C}$ and $\Re(\alpha),\Re(\beta)>0$, then the Euler transform of the function $f(t)$ is defined by

$$B\{f(t);\alpha,\beta\}=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}f(t)dt. \tag{1.9}$$

Definition 1.2. Laplace Transform: The Laplace transform of the function $f(t)$ is defined as

$$F(s)=L\{f(t);s\}=\int_0^\infty e^{-st}f(t)dt, \quad \Re(s)>0. \tag{1.10}$$

Definition 1.3. *K*-Transform: The *K*-transform with p as a complex parameter defined by

$$R_v\{f(x);p\}=g(p;v)=\int_0^\infty (px)^{\frac{1}{2}}K_v(px)f(x)dx. \tag{1.11}$$

Also, we need the following formula (see [13, page 78])

$$\int_0^\infty x^{\rho-1}K_v(ax)dx=2^{\rho-2}a^{-\rho}\Gamma\left(\frac{\rho\pm v}{2}\right). \tag{1.12}$$

Definition 1.4. Whittaker Transform: The integral transform

$$F(x)=\int_0^\infty (2xt)^{-\frac{1}{4}}W_{\lambda,\mu}(2xt)f(t)dt, \tag{1.13}$$

where $W_{\lambda,\mu}$ is the Whittaker function [12].

2 Main results

In this section, we give some integral transforms (for example, Euler, Laplace, Whittaker etc.) of generalized *k*-Bessel function given in (1.3).

Theorem 2.1. *If* $k\in\mathbb{R};\alpha,\beta,\sigma,\gamma,v,c,b\in\mathbb{C};\Re(\beta)>0,\Re(\sigma)>0,\Re(\lambda)>0,\Re\left(\frac{\gamma}{k}\right)>0,\Re\left(\frac{v}{k}+\frac{b+1}{k}\right)>0$, $\min\{\Re(\sigma v+\alpha),\Re(\sigma v+\beta)>0\}>0$, *then*

$$\begin{aligned}&\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}J_{k,v}^{b,c,\gamma,\lambda}(xz^\sigma)dz\\&=\frac{\Gamma(\beta)\left(\frac{x}{2}\right)^v k^{1-\frac{v}{k}-\frac{b+1}{2k}}}{\Gamma\left(\frac{\gamma}{k}\right)}{}_2\Psi_3\left[\begin{matrix}\left(\frac{\gamma}{k},1\right),(\sigma v+\alpha,2\sigma)\\ \left(\frac{v}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(\sigma v+\beta,2\sigma)\end{matrix}\,\middle|\,\frac{cx^2}{4k^{\frac{\lambda}{k}-1}}\right],\end{aligned} \tag{2.1}$$

where ${}_2\Psi_3$ *is the Wright hypergeometric function defined by* (1.7).

Proof. In order to derive (2.1), we denote the L.H.S. of (2.1) by I_1 and then by using the definition of generalized k-Bessel function given in (1.3), we have

$$\mathcal{I}_1=\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{(xz^{\sigma}/2)^{v+2n}}{(n!)^2}dz.$$

Interchanging the integration and summation with suitable convergence conditions, we get

$$\mathcal{I}_1=\sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{x}{2}\right)^{v+2n}}{(n!)^2}\int_0^1 z^{\alpha+\sigma v+2\sigma n-1}(1-z)^{\beta-1}dz,$$

which upon using the definition of Beta function gives

$$\mathcal{I}_1=\sum_{n=0}^{\infty}\frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{x}{2}\right)^{v+2n}}{(n!)^2}\frac{\Gamma(\alpha+2\sigma n+\sigma v)\Gamma(\beta)}{\Gamma(\alpha+\beta+2\sigma n+\sigma v)}.$$

Now using (1.4) and (1.5), we get

$$\begin{aligned}\mathcal{I}_1=&\Gamma(\beta)\left(\frac{x}{2}\right)^{v}\sum_{n=0}^{\infty}\frac{(c)^n k^n\Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\lambda n+\mu}{k}+\frac{l+1}{2k}-1}\Gamma\left(\frac{\lambda}{k}n+\frac{v}{k}+\frac{b+1}{2k}\right)}\\&\times\frac{\Gamma(\alpha+2\sigma n+\sigma v)}{\Gamma(\alpha+\beta+2\sigma n+\sigma v)}\frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2}.\end{aligned}$$

In view of the definition of (1.7) we get the desired result. □

Corollary 2.1. If we take $k=1$, in Theorem 2.1, then we have the following integral transform

$$\begin{aligned}&\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}J_{1,v}^{b,c,\gamma,\lambda}(xz^{\sigma})dz\\=&\frac{\Gamma(\beta)\left(\frac{x}{2}\right)^{v}}{\Gamma(\gamma)}\,{}_2\Psi_3\left[\begin{array}{c}(\gamma,1),(\sigma v+\alpha,2\sigma)\\\left(v+\frac{b+1}{2},\lambda\right),(\sigma v+\beta,2\sigma)\end{array}\middle|\frac{cx^2}{4}\right].\end{aligned}$$

Theorem 2.2. *If* $k\in\mathbb{R}$, $b,\lambda,v,\gamma,a,c,\sigma\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(v)\}>0$, $\Re(s)>0$, $\Re\left(\frac{\gamma}{k}\right)>0$, $\Re(\sigma v+a)>0$, $\Re\left(\frac{v}{k}+\frac{b+1}{2k}\right)>0$ *and* $\left|\frac{x}{s^{\sigma}}\right|<1$, *then we have*

$$\begin{aligned}&\int_0^{\infty} z^{a-1}e^{-sz}J_{k,v}^{b,c,\gamma,\lambda}(xz^{\sigma})dz\\=&\frac{\left(\frac{x}{2}\right)^{v}s^{-(v\sigma+a)}k^{1-\frac{v}{k}-\frac{b+1}{2k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_2\left[\begin{array}{c}\left(\frac{\gamma}{k},1\right),(\sigma v+a,2\sigma)\\\left(\frac{v}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(1,1)\end{array}\middle|\frac{cx^2k^{1-\frac{\lambda}{k}}}{4s^{2\sigma}}\right].\end{aligned}\tag{2.2}$$

Proof. Let $\mathcal{I}_2$ denoted by the left hand side of Theorem 2.2. Applying (1.3) on the L.H.S. of (2.2), to get

$$\mathcal{I}_2=\int_0^\infty z^{a-1}e^{-sz}\sum_{n=0}^\infty \frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{xz^\sigma}{2}\right)^{v+2n}}{(n!)^2}dz.$$

Interchanging the integration and summation allow us to write,

$$\mathcal{I}_2=\sum_{n=0}^\infty \frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{x}{2}\right)^{v+2n}}{(n!)^2}\int_0^1 z^{a+\sigma v+2\sigma n-1}e^{-sz}dz.$$

In view of the definition of Laplace transform, we get

$$\mathcal{I}_2=\sum_{n=0}^\infty \frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{x}{2}\right)^{v+2n}}{(n!)^2}\frac{\Gamma(a+2\sigma n+\sigma v)}{s^{a+2\sigma n+\sigma v}}.$$

Now by applying (1.5) in the above equation, we have

$$\mathcal{I}_2=\left(\frac{x}{2}\right)^v s^{-(v\sigma+a)}\sum_{n=0}^\infty \frac{(c)^n k^n\Gamma\left(\frac{\gamma}{k}+n\right)\Gamma(a+2\sigma n+\sigma v)}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\lambda n+v}{k}+\frac{b+1}{2k}}\Gamma\left(\frac{\lambda n+v}{k}+\frac{b+1}{2k}\right)}\frac{\left(\frac{x}{2}\right)^{2n}}{(n!)^2},$$

which by using the definition (1.7), gives our desired result. □

Corollary 2.2. For $k=1$ in Theorem 2.2, we have

$$\begin{aligned}&\int_0^\infty z^{a-1}e^{-sz}J_{k,v}^{b,c,\gamma,\lambda}(xz^\sigma)dz\\ =&\frac{\left(\frac{x}{2}\right)^v s^{-(v\sigma+a)}}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{c}(\gamma,1),(\sigma v+a,2\sigma)\\ \left(v+\frac{b+1}{2},\lambda\right),(1,1)\end{array}\middle|\frac{cx^2}{4s^{2\sigma}}\right].\end{aligned}$$

Theorem 2.3. *If* $k\in\mathbb{R}$, b, $\lambda,v,\gamma,a,c,\sigma\in\mathbb{C}$, $\min\{\Re(\lambda),\Re(\sigma)\}>0$, $\Re(\rho)>0$, $\Re\left(\frac{\gamma}{k}\right)>0$, $\Re(\sigma v+a)>0$, $\Re\left(\frac{\rho+\sigma v\pm\mu}{2}\right)>0$ *and* $\left|\frac{x}{s^\sigma}\right|<1$, *then*

$$\begin{aligned}&\int_0^\infty z^{\rho-1}K_\mu(az)J_{k,v}^{b,c,\gamma,\lambda}(xz^\sigma)dz\\ =&\frac{\left(\frac{x}{2}\right)^v 2^{(\rho+\sigma v-2)}a^{-(\rho+\sigma v)}k^{1-\frac{v}{k}-\frac{b+1}{2k}}}{\Gamma\left(\frac{\gamma}{k}\right)}\,{}_2\Psi_2\left[\begin{array}{c}\left(\frac{\gamma}{k},1\right),\left(\frac{\rho+\sigma v\pm\mu}{2},\sigma\right)\\ \left(\frac{v}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),(1,1)\end{array}\middle|\frac{c4^\sigma x^2k^{1-\frac{\lambda}{k}}}{4a^{2\sigma}}\right].\end{aligned}\tag{2.3}$$

Proof. Let $\mathcal{I}_3$ denoted by the left hand side of Theorem 2.3. Applying (1.3) on the L.H.S. of (2.3), to get

$$\mathcal{I}_3=\int_0^\infty z^{\rho-1}K_\mu(az)\sum_{n=0}^\infty \frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{xz^\sigma}{2}\right)^{v+2n}}{(n!)^2}dz.$$

Interchanging the integration and summation allow us to write,

$$\mathcal{I}_3=\sum_{n=0}^\infty \frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{x}{2}\right)^{v+2n}}{(n!)^2}\int_0^1 z^{\rho+\sigma v+2\sigma n-1}K_\mu(az)dz.$$

Using the formula given in (1.12) in the above expression, we get

$$\mathcal{I}_3=\sum_{n=0}^\infty \frac{(c)^n(\gamma)_{n,k}}{\Gamma_k\left(\lambda n+v+\frac{b+1}{2}\right)}\frac{\left(\frac{x}{2}\right)^{v+2n}}{(n!)^2}2^{(\rho+2\sigma n+\sigma v-2)}a^{-(\rho+2\sigma n+\sigma v)}\Gamma\left(\frac{\rho+2\sigma n+\sigma v\pm\mu}{2}\right).$$

By applying (1.5), we have

$$\mathcal{I}_3=\left(\frac{x}{2}\right)^v 2^{(\rho+\sigma v-2)}a^{-(\rho+\sigma v)}\sum_{n=0}^\infty \frac{(c)^n k^n\Gamma\left(\frac{\gamma}{k}+n\right)\Gamma\left(\frac{\rho+2\sigma n+\sigma v\pm\mu}{2}\right)}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\lambda n+v}{k}+\frac{b+1}{2k}}\Gamma\left(\frac{\lambda n+v}{k}+\frac{b+1}{2k}\right)}\frac{\left(\frac{x}{2}\right)^{2n}2^{2\sigma n}a^{-2\sigma n}}{(n!)^2}.$$

In view of definition (1.7), we get the desired result. □

Corollary 2.3. For $k=1$ in Theorem 2.3, we get the following interesting integral transform

$$\int_0^\infty z^{\rho-1}K_\mu(az)J_v^{b,c,\gamma,\lambda}(xz^\sigma)dz$$
$$=\frac{\left(\frac{x}{2}\right)^v 2^{(\rho+\sigma v-2)}a^{-(\rho+\sigma v)}}{\Gamma(\gamma)}{}_2\Psi_2\left[\begin{array}{c}(\gamma,1),\left(\frac{\rho+\sigma v\pm\mu}{2},\sigma\right)\\(v+b+1,\lambda),(1,1)\end{array}\middle|\frac{c4^\sigma x^2}{4a^{2\sigma}}\right].$$

In the following theorem, we derive the Whittaker transform of generalized k-Bessel function. Here, we recall the following results:

$$\int_0^\infty t^{v-1}e^{-\frac{t}{2}}W_{\lambda,\mu}(t)dt=\frac{\Gamma\left(\frac{1}{2}+\mu+v\right)\Gamma\left(\frac{1}{2}-\mu+v\right)}{\Gamma(1-\lambda+v)},\quad \Re(v\pm\mu)>-1/2, \tag{2.4}$$

where the Whittaker function $W_{\lambda,\mu}(t)$ is given in [12] (also see [14]), and

$$\int_0^\infty t^{\nu-1}e^{-\frac{t}{2}}M_{\lambda,m}(t)dt=\frac{\Gamma(2m+1)\Gamma(m+\nu+\frac{1}{2})\Gamma(\lambda-\nu)}{\Gamma(m-\nu+\frac{1}{2})\Gamma(m+\lambda+\frac{1}{2})}, \tag{2.5}$$

where $M_{\lambda,\mu}(t)$ is defined as

$$M_{\lambda,\mu}(t)=z^{\frac{1}{2}+\mu}e^{-\frac{1}{2}t}\,{}_1F_1\left(\frac{1}{2}+\mu+\lambda;2\mu+1;t\right).$$

Theorem 2.4. *If* $k\in\mathbb{R}^+$, $\lambda,\mu,\beta,b,c,\gamma,\lambda\in\mathbb{C}$, $\Re(\rho\pm\mu)>-\frac{1}{2}$, $\Re(\rho)>0$, $\Re\left(\frac{\gamma}{k}\right)>0$, $\Re\left(\frac{\sigma}{k}+\frac{l+1}{2k}\right)>0$, $\min\{\Re(\delta),\Re(\lambda)\}>0$, $\Re(\rho-\lambda')>0$, *then*

$$\begin{aligned}&\int_0^\infty t^{\rho-1}e^{-\frac{pt}{2}}W_{\lambda',\mu}(pt)\,J_{k,\sigma}^{b,c,\gamma,\lambda}\left(wt^\delta\right)dt\\ =&\frac{1}{p^{\rho+\sigma\delta}}\frac{(w/2)^\sigma}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\sigma}{k}+\frac{b+1}{2k}-1}}\\ &\times{}_3\Psi_3\left[\begin{array}{c}\left(\frac{\gamma}{k},1\right),\left(\frac{1}{2}+\mu+\rho+\delta\sigma,2\delta\right),\left(\frac{1}{2}-\mu+\rho+\delta\sigma,2\delta\right)\\ \left(\frac{\sigma}{k}+\frac{l+1}{2k},\frac{\lambda}{k}\right),(1-\lambda'+\rho+\delta\sigma,2\delta),(1,1)\end{array}\middle|\frac{ck^{1-\frac{\lambda}{k}}w^2}{4p^{2\delta}}\right],\end{aligned}\tag{2.6}$$

where $W_{\lambda',\mu}$ *is the Whittaker function of second kind (see [1]).*

Proof. Let $pt=v$, then

$$\begin{aligned}&\int_0^\infty t^{\rho-1}e^{-\frac{pt}{2}}W_{\lambda',\mu}(pt)\,J_{k,\sigma}^{b,c,\gamma,\lambda}\left(wt^\delta\right)dt\\ =&\int_0^\infty e^{-\frac{v}{2}}\left(\frac{v}{p}\right)^{\rho-1}W_{\lambda',\mu}(v)\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\sigma}}{(n!)^2\Gamma_k\left(\lambda n+\sigma+\frac{b+1}{2}\right)}\left[\left(\frac{\vartheta}{p}\right)^\delta\right]^{2n+\sigma}\frac{1}{p}dv.\end{aligned}$$

Interchanging the integration and summation allow us to write

$$\begin{aligned}&=\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\sigma}}{(n!)^2\Gamma_k\left(\lambda n+\sigma+\frac{b+1}{2}\right)p^{2\delta n}}\int_0^\infty e^{-\frac{v}{2}}\left(\frac{v}{p}\right)^{\rho+2n\delta+\sigma\delta-1}W_{\lambda',\mu}(v)\frac{1}{p}dv,\\ &=\frac{1}{p^{\rho+\sigma\delta}}\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\sigma}}{(n!)^2\Gamma_k\left(\lambda n+\sigma+\frac{b+1}{2}\right)p^{2\delta n}}\int_0^\infty e^{-\frac{v}{2}}(v)^{\rho+2n\delta+\sigma\delta-1}W_{\lambda',\mu}(v)dv.\end{aligned}$$

Using the formula for Whittaker transform (2.4), we get

$$\begin{aligned}=&\frac{1}{p^{\rho+\sigma\delta}}\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\sigma}}{(n!)^2\Gamma_k\left(\lambda n+\sigma+\frac{b+1}{2}\right)p^{2\delta n}}\\ &\times\frac{\Gamma\left(\frac{1}{2}+\mu+2\delta n+\rho+\sigma\delta\right)\Gamma\left(\frac{1}{2}-\mu+2\delta n+\rho+\sigma\delta\right)}{\Gamma(1-\lambda'+\rho+2\delta n+\sigma\delta)}.\end{aligned}$$

Applying (1.4) and (1.5), to get

$$\begin{aligned}=&\frac{1}{p^{\rho+\sigma\delta}}\frac{(w/2)^\sigma}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\sigma}{k}+\frac{b+1}{2k}-1}}\sum_{n=0}^\infty\frac{c^nk^n\Gamma\left(\frac{\gamma}{k}+n\right)\left(\frac{w^2}{4p^{2\delta}}\right)^n}{k^{\frac{\lambda}{k}n}(n!)^2\Gamma\left(\frac{\lambda n+\sigma}{k}+\frac{b+1}{2k}\right)}\\ &\times\frac{\Gamma\left(\frac{1}{2}+\mu+2\delta n+\rho+\sigma\delta\right)\Gamma\left(\frac{1}{2}-\mu+2\delta n+\rho+\sigma\delta\right)}{\Gamma(1-\lambda'+\rho+2\delta n+\sigma\delta)}.\end{aligned}$$

In view of (1.7), we get the desired result. □

Corollary 2.4. For $k=1$ in Theorem 2.4, we get

$$\int_0^\infty t^{\rho-1}e^{-\frac{pt}{2}}W_{\lambda,\mu}(pt)J_{1,\sigma}^{b,c,\gamma,\lambda}\left(wt^\delta\right)dt$$
$$=\frac{1}{p^{\rho+\sigma\delta}}\frac{(w/2)^\sigma}{\Gamma(\gamma)}\,{}_3\Psi_3\left[\begin{array}{c}(\gamma,1),\left(\frac{1}{2}+\mu+\rho+\delta\sigma,2\delta\right),\left(\frac{1}{2}-\mu+\rho+\delta\sigma,2\delta\right)\\(\sigma+\sigma,\lambda),(1-\lambda'+\rho+\delta\sigma,2\delta),(1,1)\end{array}\middle|\frac{cw^2}{4p^{2\delta}}\right].$$

Theorem 2.5. *If* $k\in\mathbb{R}^+$, $\lambda,\mu,\beta,b,c,\gamma,\lambda\in\mathbb{C}$, $\Re\left(\frac{\gamma}{k}\right)>0$, $\Re\left(\frac{\sigma}{k}+\frac{l+1}{2k}\right)>0$, $\min\{\Re(\delta),\Re(\lambda)\}>0$, $\Re(m)>-\frac{1}{2}$, $\Re(\mu)>0$, $\Re(\nu)>0$ *and* $\Re(m\pm\rho)>-\frac{1}{2}$, *then*

$$\int_0^\infty t^{\rho-1}e^{-\frac{pt}{2}}M_{\mu,m}(pt)J_{k,\nu}^{b,c,\gamma,\lambda}\left(wt^\delta\right)dt$$
$$=\frac{1}{p^{\rho+\nu\delta}}\frac{(w/2)^\nu\Gamma(2m+1)}{\Gamma\left(\frac{\gamma}{k}\right)\Gamma(m+\mu+\frac{1}{2})k^{\frac{\nu}{k}+\frac{b+1}{2k}-1}}$$
$$\times{}_3\Psi_3\left[\begin{array}{c}\left(\frac{\gamma}{k},1\right),\left(\frac{1}{2}+m+\rho+\nu\delta,2\delta\right),(\mu-\rho-\nu\delta,-2\delta)\\\left(\frac{\nu}{k}+\frac{b+1}{2k},\frac{\lambda}{k}\right),\left(m-\rho-\nu\delta+\frac{1}{2},-2\delta\right),(1,1)\end{array}\middle|\frac{ck^{1-\frac{\lambda}{k}}w^2}{4p^{2\delta}}\right]. \tag{2.7}$$

Proof. Let $pt=v$, then

$$\int_0^\infty t^{\rho-1}e^{-\frac{pt}{2}}M_{\mu,m}(pt)J_{k,\nu}^{b,c,\gamma,\lambda}\left(wt^\delta\right)dt$$
$$=\int_0^\infty e^{-\frac{v}{2}}\left(\frac{v}{p}\right)^{\rho-1}M_{\mu,m}(v)\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\nu}}{(n!)^2\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)}\left[\left(\frac{\vartheta}{p}\right)^\delta\right]^{2n+\nu}\frac{1}{p}dv.$$

Interchanging the integration and summation allow us to write

$$=\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\nu}}{(n!)^2\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)p^{2\delta n}}\int_0^\infty e^{-\frac{v}{2}}\left(\frac{v}{p}\right)^{\rho+2n\delta+\nu\delta-1}M_{\mu,m}(v)\frac{1}{p}dv$$
$$=\frac{1}{p^{\rho+\nu\delta}}\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\nu}}{(n!)^2\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)p^{2\delta n}}\int_0^\infty e^{-\frac{v}{2}}(v)^{\rho+2n\delta+\nu\delta-1}M_{\mu,m}(v)dv.$$

On using (2.5), we get

$$=\frac{1}{p^{\rho+\nu\delta}}\sum_{n=0}^\infty\frac{c^n(\gamma)_{n,k}(w/2)^{2n+\nu}}{(n!)^2\Gamma_k\left(\lambda n+\nu+\frac{b+1}{2}\right)p^{2\delta n}}$$
$$\times\frac{\Gamma(2m+1)\Gamma\left(m+\rho+\nu\delta+2n\delta+\frac{1}{2}\right)\Gamma(\mu-\rho-\nu\delta-2n\delta)}{\Gamma\left(m-\rho-\nu\delta-2n\delta+\frac{1}{2}\right)\Gamma\left(m+\mu+\frac{1}{2}\right)}.$$

By applying (1.4) and (1.5), we obtain

$$=\frac{1}{p^{\rho+\nu\delta}}\frac{(w/2)^{\nu}}{\Gamma\left(\frac{\gamma}{k}\right)k^{\frac{\nu}{k}+\frac{b+1}{2k}-1}}\sum_{n=0}^{\infty}\frac{c^n k^n\Gamma\left(\frac{\gamma}{k}+n\right)\left(\frac{w^2}{4p^{2\delta}}\right)^n}{k^{\frac{\lambda}{k}n}(n!)^2\Gamma\left(\frac{\lambda n+\nu}{k}+\frac{b+1}{2k}\right)}$$
$$\times\frac{\Gamma(2m+1)\Gamma\left(m+\rho+\nu\delta+2n\delta+\frac{1}{2}\right)\Gamma(\mu-\rho-\nu\delta-2n\delta)}{\Gamma\left(m-\rho-\nu\delta-2n\delta+\frac{1}{2}\right)\Gamma\left(m+\mu+\frac{1}{2}\right)}.$$

In view of (1.7), we get the desired result. □

Corollary 2.5. For $k=1$ in Theorem 2.5, we get

$$\int_0^{\infty}t^{\rho-1}e^{-\frac{pt}{2}}M_{\mu,m}(pt)J_{1,\nu}^{b,c,\gamma,\lambda}\left(wt^{\delta}\right)dt$$
$$=\frac{1}{p^{\rho+\sigma\delta}}\frac{(w/2)^{\sigma}}{\Gamma(\gamma)}\,{}_3\Psi_3\left[\begin{array}{c}(\gamma,1),\left(\frac{1}{2}+\mu+\rho+\delta\sigma,2\delta\right),\left(\frac{1}{2}-\mu+\rho+\delta\sigma,2\delta\right)\\(\sigma+\sigma,\lambda),(1-\lambda+\rho+\delta\sigma,2\delta),(1,1)\end{array}\middle|\frac{cw^2}{4p^{2\delta}}\right].$$

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