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Some Inequalities Concerning the Polar Derivative of a Polynomial-II

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Abstract. In this paper, we consider the class of polynomials $P(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \le \mu \le n$, having all zeros in $|z| \le k$, $k \le 1$ and thereby present an alternative proof, independent of Laguerre's theorem, of an inequality concerning the polar derivative of a polynomial.

Key Words: Polar derivative of a polynomial.

AMS Subject Classifications: 30A10, 30C10, 30C15

1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative, then according to the well-known Bernstein's inequality [2] on the derivative of a polynomial, we have

$$Max_{|z|=1}|P'(z)| \le n Max_{|z|=1}|P(z)|. \tag{1.1}$$

The equality (1.1) is best possible and equality holds if and only if $P(z)$ has all its zeros at the origin.

For the class of polynomials $P(z)$ of degree n having all zeros in $|z| \le 1$, Turan [7] proved that

$$Max_{|z|=1}|P'(z)| \ge \frac{n}{2} Max_{|z|=1}|P(z)|. \tag{1.2}$$

The inequality (1.2) is best possible and become equality for polynomials having all zeros on $|z| = 1$.

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As an extension of (1.2), Malik [6] proved that if $P(z)$ has all its zeros in $|z|\leq k$, $k\leq 1$, then

$$Max_{|z|=1}|P'(z)|\geq \frac{n}{1+k}Max_{|z|=1}|P(z)|. \tag{1.3}$$

As a refinement of (1.3), Govil [5] under the same hypothesis proved that

$$Max_{|z|=1}|P'(z)|\geq \frac{n}{1+k}\Big\{Max_{|z|=1}|P(z)|+\frac{1}{k^{n-1}}Min_{|z|=k}|P(z)|\Big\}. \tag{1.4}$$

Aziz and Shah [1] generalized (1.4) in a different direction and proved that, if $P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $\mu\geq 1$, is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then

$$Max_{|z|=1}|P'(z)|\geq \frac{n}{1+k^{\mu}}\Big\{Max_{|z|=1}|P(z)|+\frac{1}{k^{n-\mu}}Min_{|z|=k}|P(z)|\Big\}. \tag{1.5}$$

For $\mu=1$, inequality (1.5) reduces to inequality (1.4).

Let $D_\alpha P(z)$ denotes the polar derivative of the polynomial $P(z)$ of degree n with respect to the point $\alpha\in C$. Then

$$D_\alpha P(z)=nP(z)+(\alpha-z)P'(z).$$

The polynomial $D_\alpha P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\Big[\frac{D_\alpha P(z)}{\alpha}\Big]=P'(z).$$

Dewan, Singh and Lal [4] extend the inequality (1.5) to the polar derivative of a polynomial $P(z)$ and proved that if $P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leq\mu\leq n$, has all its zeros in $|z|\leq k$, $k\leq 1$, then for every real or complex number α with $|\alpha|\geq k^{\mu}$,

$$Max_{|z|=1}\big|D_\alpha P(z)\big|\geq n\Big(\frac{|\alpha|-k^{\mu}}{1+k^{\mu}}\Big)Max_{|z|=1}|P(z)|+\frac{n(|\alpha|+1)}{k^{n-\mu}(1+k^{\mu})}Min_{|z|=k}|P(z)|. \tag{1.6}$$

As a refinement of (1.6), Dewan, Singh and Mir [3] proved the following result:

Theorem 1.1. *Let $P(z)=a_nz^n+\sum_{\nu=\mu}^{n}a_{n-\nu}z^{n-\nu}$, $1\leq\mu\leq n$, be a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for every real or complex number α with $|\alpha|\geq k^{\mu}$, we have*

$$Max_{|z|=1}\big|D_\alpha P(z)\big|\geq n\Big(\frac{|\alpha|-A_\mu}{1+k^{\mu}}\Big)Max_{|z|=1}|P(z)|+\frac{n}{k^n}\Big(\frac{|\alpha|k^{\mu}+A_\mu}{1+k^{\mu}}\Big)Min_{|z|=k}|P(z)|,$$

where

$$A_\mu=\frac{n\big(|a_n|-\frac{m}{k^n}\big)k^{2\mu}+\mu|a_{n-\mu}|k^{\mu-1}}{n\big(|a_n|-\frac{m}{k^n}\big)k^{\mu-1}+\mu|a_{n-\mu}|}. \tag{1.7}$$