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Uncertainty Principles for the Generalized Fourier Transform Associated with Spherical Mean Operator

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Abstract. The aim of this paper is to establish an extension of qualitative and quantitative uncertainty principles for the Fourier transform connected with the spherical mean operator.

Key Words: Generalized Fourier transform, Hardy's type theorem, Cowling-Price's theorem, Beurling's theorem, Miyachi's theorem, Donoho-Stark's uncertainty principle.
AMS Subject Classifications: 43A32, 42B10

1 Introduction

Classical uncertainty principles give us information about a function and its Fourier transform. If we try to limit the behavior of one we lose control of the other. Uncertainty principles have implications in two main areas: quantum physics and signal analysis. In quantum physics they tell us that a particle speed and position cannot both be measured with infinite precision. In signal analysis they tell us that if we observe a signal only for a finite period of time, we will lose information about the frequencies the signal consisted of. The mathematical equivalence is that a function and its Fourier transform cannot both be arbitrarily localized. There is two categories of uncertainty principles: Quantitative uncertainty principles and Qualitative uncertainty principles.

Quantitative uncertainty principle is just another name for some special inequalities. These inequalities give us information about how a function and its Fourier transform relates. They are called uncertainty principles since they are similar to the classical Heisenberg Uncertainty Principle, which has had a big part to play in the development and

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understanding of quantum physics. For example: Benedicks [2], Slepian and Pollak [26], Landau and Pollak [17], and Donoho and Stark [9] paid attention to the supports of functions and gave qualitative uncertainty principles for the Fourier transforms.

Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [13], Morgan [21], Cowling and Price [7], Beurling [3], Miyachi [20] theorems enter within the framework of the quantitative uncertainty principles.

The quantitative and qualitative uncertainty principles has been studied by many authors for various Fourier transforms, for examples (cf. [5, 6, 11, 12, 18, 19, 27]).

Our aim here is to consider quantitative and qualitative uncertainty principles when the transform under consideration is the Fourier transform connected with the spherical mean operator. The spherical mean operator play an important role and have many applications, for example; in the image processing of so-called synthetic aperture radar (SAR) data [14, 15], or in the linearized inverse scattering problem in acoustics [10]. These operators have been studied by many authors from many points of view [1, 10, 22, 24].

The remaining part of the paper is organized as follows. In Section 2, we recall the main results about the spherical mean operator. Section 3 is devoted to generalize Cowling-Price's theorem for the generalized Fourier transform $\mathcal{F}$. In Section 4 we generalize Miyachi's theorem and in Section 5 Beurling's theorem for $\mathcal{F}$. Section 6 is devoted to Donoho-Stark's uncertainty principle and variants of Heisenberg's inequalities for $\mathcal{F}$.

Throughout this paper, the letter C indicates a positive constant not necessarily the same in each occurrence.

2 Spherical mean operator

In this section, we define and recall some properties of the spherical mean operator. For more details see [22, 24]. We denote by

- $C_*(\mathbb{R}^{d+1})$ the space of continuous functions on $\mathbb{R}^{d+1}=\mathbb{R}\times\mathbb{R}^d$, even with respect to the first variable.
- $C_{*,c}(\mathbb{R}^{d+1})$ the subspace of $C_*(\mathbb{R}^{d+1})$ formed by functions with compact support.
- $\mathcal{E}_*(\mathbb{R}^{d+1})$ the space of infinitely differentiable functions on $\mathbb{R}^{d+1}$, even with respect to the first variable.
- S^d the unit sphere in $\mathbb{R}^{d+1}$,
$$S^d=\big\{(\eta,\xi)\in\mathbb{R}^{d+1}:\eta^2+\|\xi\|^2=1\big\},$$
where for $\xi=(\xi_1,\cdots,\xi_d)$, we have $\|\xi\|^2=\xi_1^2+\cdots+\xi_d^2$.
- $d\sigma_d$ the normalized surface measure on S^d.
- $\mathbb{R}^{d+1}_+=\big\{(r,x)\in\mathbb{R}^{d+1}:\ r>0\big\}$.