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A New Estimate for Bochner–Riesz Operators at the Critical Index on Weighted Hardy Spaces

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Abstract. Let w be a Muckenhoupt weight and $H^p_w(\mathbb{R}^n)$ be the weighted Hardy space. In this paper, by using the atomic decomposition of $H^p_w(\mathbb{R}^n)$, we will show that the Bochner–Riesz operators T^δ_R are bounded from $H^p_w(\mathbb{R}^n)$ to the weighted weak Hardy spaces $WH^p_w(\mathbb{R}^n)$ for $0<p<1$ and $\delta=n/p-(n+1)/2$. This result is new even in the unweighted case.

Key Words: Bochner–Riesz operator, weighted Hardy space, weighted weak Hardy space, A_p weight, atomic decomposition.

AMS Subject Classifications: 42B15, 42B25, 42B30

1 Introduction

The Bochner-Riesz operators of order $\delta>0$ in $\mathbb{R}^n$ are defined initially for Schwartz functions in terms of Fourier transforms by

$$\big(\widehat{T^\delta_R f}\big)(\xi)=\Big(1-\frac{|\xi|^2}{R^2}\Big)^\delta_+\hat{f}(\xi),\quad 0<R<\infty,$$

where $\hat{f}$ denotes the Fourier transform of f. The associated maximal Bochner-Riesz operator is defined by

$$T^\delta_* f(x)=\sup_{R>0}\big|T^\delta_R f(x)\big|.$$

These operators were first introduced by Bochner [1] in connection with summation of multiple Fourier series and played an important role in harmonic analysis. The problem concerning the spherical convergence of Fourier integrals has led to the study of its L^p boundedness. As for its H^p boundedness, Sjölin [17] and Stein, Taibleson and Weiss [18] proved the following theorem (see also [11, pp. 130]).

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Theorem 1.1. *Suppose that $0<p\le 1$ and $\delta>n/p-(n+1)/2$. Then there exists a constant $C>0$ independent of f and R such that*

$$\big\|T_R^\delta(f)\big\|_{H^p}\le C\|f\|_{H^p}.$$

In [18], the authors also considered the weak type estimate for the maximal Bochner-Riesz operator T_*^δ at the critical index $\delta=n/p-(n+1)/2$ and showed the following inequality is sharp.

Theorem 1.2. *Suppose that $0<p<1$ and $\delta=n/p-(n+1)/2$. Then there exists a constant $C>0$ independent of f such that*

$$\sup_{\lambda>0}\lambda^p\big|\big\{x\in\mathbb{R}^n: T_*^\delta f(x)>\lambda\big\}\big|\le C\|f\|_{H^p}^p.$$

In 1995, Sato [15] studied the weighted case and obtained the following weighted weak type estimate for the maximal Bochner-Riesz operator T_*^δ.

Theorem 1.3. *Let $w\in A_1$ (Muckenhoupt weight class), $0<p<1$ and $\delta=n/p-(n+1)/2$. Then there exists a constant $C>0$ independent of f such that*

$$\sup_{\lambda>0}\lambda^p\cdot w\big(\big\{x\in\mathbb{R}^n: T_*^\delta f(x)>\lambda\big\}\big)\le C\|f\|_{H_w^p}^p.$$

In [16], Sato also showed that for $n\ge 2$, there exists a function $f\in H_w^1\cap L^1$, $w\in A_1$, such that

$$\limsup_{R\to\infty}\big|T_R^{(n-1)/2}f(x)\big|=+\infty \text{ almost everywhere.}$$

Hence, $\delta=n/p-(n+1)/2$ is indeed the critical index for the weighted case.

In 2006, Lee [8] considered values of δ greater than the critical index $n/p-(n+1)/2$ and proved $H_w^p-L_w^p$ boundedness of the maximal operator T_*^δ. Furthermore, by using this strong type estimate of T_*^δ, Lee [8] also obtained H_w^p boundedness of the Bochner-Riesz operator.

The purpose of this article is to discuss the boundedness of Bochner-Riesz operators at the critical index $n/p-(n+1)/2$ on weighted Hardy spaces. Our main result is stated as follows.

Theorem 1.4. *Let $0<p<1$, $\delta=n/p-(n+1)/2$ and $w\in A_1$. Suppose that $n(1/p-1)$ is not a positive integer, then there exists a constant $C>0$ independent of f and R such that*

$$\big\|T_R^\delta(f)\big\|_{WH_w^p}\le C\|f\|_{H_w^p},$$

where WH_w^p denotes the weighted weak Hardy space.

In particular, if we take w to be a constant function, then we can get

Corollary 1.1. Let $0<p<1$ and $\delta=n/p-(n+1)/2$. Suppose that $n(1/p-1)$ is not a positive integer, then there exists a constant $C>0$ independent of f and R such that

$$\big\|T_R^\delta(f)\big\|_{WH^p}\le C\|f\|_{H^p},$$

where WH^p denotes the weak Hardy space.