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Two Weighted *BMO* Estimates for the Maximal Bochner-Riesz Commutator

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Abstract. In this note, the author prove that maximal Bocher-Riesz commutator $B^b_{\delta,*}$ generated by operator $B_{\delta,*}$ and function $b\in BMO(\omega)$ is a bounded operator from $L^p(\mu)$ into $L^p(\nu)$, where $\omega\in(\mu\nu^{-1})^{\frac{1}{p}},\mu,\nu\in A_p$ for $1<p<\infty$. The proof relies heavily on the pointwise estimates for the sharp maximal function of the commutator $B^b_{\delta,*}$.

Key Words: Bocher-Riesz operator, commutator, weighted $BMO(\omega)$ space.

AMS Subject Classifications: 42B25, 42B30

1 Introduction

It is well-known that the commutator is an important integral operator and plays a key role in harmonic analysis. In 1965, Calderón [1, 2] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Subsequently, Coifman, Rochberg and Weiss [3] obtained the boundedness of singular integral commutators on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. In 1995, Lu [4] gave the definition of the maximal Bochner-Riesz operator and its maximal commutator as the following.

Definition 1.1. Let $\widehat{B^\delta_t(f)}(\xi)=(1-t^2|\xi|^2)^\delta\hat{f}(\xi)$ and $\psi^\delta_t(z)=t^{-n}\psi^\delta(\frac{z}{t})$ for $t>0$. We denote

$$B^b_{\delta,t}(f)(x)=\int_{R^n}[b(x)-b(y)]\psi^\delta_t(x-y)f(y)dy.$$

The maximal Bochner-Riesz operator and maximal commutator are defined respectively by

$$B^\delta_*(f)(x)=\sup_{t>0}|\psi^\delta_t*f(x)|,$$

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and

$$B_{\delta,*}^b(f)(x)=\sup_{t>0}|B_{\delta,t}^b(f)(x)|.$$

In 1996, Lu and Hu [4,5] established the boundedness of Bochner-Riesz commutator on $L^p(\mathbb{R}^n)$ $(1<p<\infty)$. In 1997, Yang and Lu [6] studied the continuity of Bochner-Riesz commutator on Herz-type spaces. In 2003, Lu and Liu [7] established the $L(\log L)$ type estimate and weighted weak type estimate of Bochner-Riesz maximal commutator. In 2004, Liu [7] established the continuity of Bochner-Riesz maximal commutator on Triebel-Lizorkin space. The main purpose of this paper is to give the two-weighted estimate of the maximal Bochner-Riesz commutator.

Our main result is stated as follows.

Theorem 1.1. *Let $B_{\delta,*}^b$ be defined as before, and $b\in BMO(\omega)$, $\omega=(\mu\nu^{-1})^{\frac{1}{p}}$, $\mu(x),\nu(x)\in A_p$. If $\delta>\frac{n-1}{2}$, then there exists a positive constant C such that*

$$\|B_{\delta,*}^b(f)(x)\|_{L^p(\nu)}\le C\|b\|_{*,\omega}\|f\|_{L^p(\mu)}.$$

2 Some preliminaries and main results

Standard real analysis tools as the maximal function M,f the sharp function $M^\sharp,f$ naturally carries over to this context. Let $B=B(x,r)$, $B(x,kr)=kB$, Define the maximal functions

$$\begin{aligned} Mf(x)&=\sup_{x\in B}\frac{1}{|B|}\int_B|f(y)|dy,\\ M^\sharp f(x)&=\sup_{x\in B}\frac{1}{|B|}\int_B|f(y)-f_B|dy\approx\sup_{x\in B}\inf_C\frac{1}{|B|}\int_B|f(y)-C|dy, \end{aligned}$$

where $f_B=\frac{1}{|B|}\int_B|f(y)|dy$.

Let ω be a weight function, we will say that a locally integrable function $b(x)$ belongs to the weighted $BMO(\omega)$ space for $\omega\in A_p$, if

$$\|b\|_{*,\omega}=\sup_B\frac{1}{\omega(B)}\int_B|b(x)-b_B|dx<\infty,$$

where the supremum is taken over all balls $B\subset R^n$. Obviously, for the case ω being the Lebesgue measure, $BMO(\omega)=BMO$.

A variant of the maximal operator

$$M_\sigma f(x)=(M(|f|^\sigma))^{\frac{1}{\sigma}}(x),$$

and

$$M_\sigma^\sharp f(x)=(M^\sharp(|f|^\sigma))^{\frac{1}{\sigma}}(x)$$

will become the main tool in our scheme.