Boundedness for Multilinear Commutators of Marcinkiewicz Integral on Morrey-Herz Spaces with Non Doubling Measures

Jianglong Wu\textsuperscript{1,*} and Qingguo Liu\textsuperscript{2}

1 Department of Mathematics, Mudanjiang Normal University, Mudanjiang 157011, Hei Longjiang, China
2 University of Nova Gorica, Nova Gorica 5000, Slovenia

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Abstract. In this paper, the authors establish the boundedness of multilinear commutators generated by a Marcinkiewicz integral operator and a RBMO(μ) function on homogeneous Morrey-Herz spaces with non doubling measures.

Key Words: Marcinkiewicz integral, commutator, Morrey-Herz space, non doubling measure, RBMO function.

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1 Introduction and preliminaries

As an analogy of the classical Littlewood-Paley g function, Marcinkiewicz [1] introduced the operator

\[ M(f)(x) = \left( \int_{0}^{\pi} \left\| F(x+t) + F(x-t) - 2F(x) \right\|^2 t^3 \, dt \right)^{\frac{1}{2}}, \quad x \in [0,2\pi], \]

where \( F(x) = \int_{0}^{x} f(t) \, dt \). This operator is now called the Marcinkiewicz integral. Zygmund [2] proved that the operator \( M \) is bounded on the Lebesgue space \( L^p([0,2\pi]) \) for \( p \in (1,\infty) \). Stein [3] generalized the above Marcinkiewicz integral to the following higher-dimensional case. Let \( \Omega \) be homogeneous of degree zero in \( \mathbb{R}^d \) for \( d \geq 2 \), integrable and have mean value zero on the unit sphere \( S^{d-1} \). The higher-dimensional Marcinkiewicz integral is defined by

\[ M_\Omega(f)(x) = \left( \int_{0}^{\infty} \left( \int_{|y| \leq t} \frac{\Omega(x-y)}{|x-y|^{d-1}} f(y) \, dy \right)^2 \frac{dt}{t^3} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^d. \]
Stein in [3] proved that if \( \Omega \in \text{Lip}_\delta(S^{d-1}) \) for some \( \delta \in (0,1] \), then \( M_\Omega \) is bounded on \( L^p(\mathbb{R}^d) \) for any \( p \in (1,2] \), and is also bounded from \( L^1(\mathbb{R}^d) \) to \( L^{1,\infty}(\mathbb{R}^d) \). Since then, a lot of papers focus on this operator. For some recent development, we mention that Al-Salman et al. in [4] obtained the \( L^p(\mathbb{R}^d) \)-boundedness for \( p \in (1,\infty) \) of \( M_\Omega \) if \( \Omega \in L(\log L)^{1/2}(S^{d-1}) \); Fan and Sato in [5] proved that \( M_\Omega \) is bounded from the Lebesgue space \( L^1(\mathbb{R}^d) \) to the weak Lebesgue space \( L^{1,\infty}(\mathbb{R}^d) \) if \( \Omega \in L\log L(S^{d-1}) \). There are many other interesting works for this operator, among them we refer to [6, 7] and their references. On the other hand, Torchinsky and Wang in [8] first introduced the commutator generated by the Marcinkiewicz integral \( M_\Omega \) and the classical \( \text{BMO}(\mathbb{R}^d) \) function, and established its \( L^p(\mathbb{R}^d) \)-boundedness for \( p \in (1,\infty) \) when \( \Omega \in \text{Lip}_\delta(S^{d-1}) \) for some \( \delta \in (0,1] \). Such boundedness of this commutator is further discussed in [9, 10] when \( \Omega \) only satisfies certain size conditions. Moreover, its weak type endpoint estimate is obtained in [11, 12] when \( \Omega \in \text{Lip}_\delta(S^{d-1}) \) for some \( \delta \in (0,1] \), and its weight weak type endpoint estimate is obtained in [13, 14] when \( \Omega \) satisfies a kind of Dini conditions. Also see [15–17] et al. for more informations.

Motivated by the work above, the main purpose of this paper is to establish a similar theory for the multilinear commutator generated by a Marcinkiewicz integral operator and a \( \text{RBMO}(\mu) \) function or \( \text{Osc}_{\exp L}(\mu) \) function on \( \mathbb{R}^d \) with a positive Radon measure which may be non doubling.

To be precise, let \( \mu \) be a positive Radon measure on \( \mathbb{R}^d \) which only satisfies the following growth condition that for all \( x \in \mathbb{R}^d \) and all \( r > 0 \),

\[
\mu(B(x,r)) \leq C_0 r^n, \tag{1.1}
\]

where \( C_0 > 0 \) and \( n \) are some positive constants, \( 0 < n \leq d \), and \( B(x,r) \) is the open ball centered at \( x \) and having radius \( r \). We recall that \( \mu \) is said to be a doubling measure, if there is a positive constant \( C \) such that for any \( x \in \text{supp} \mu \) and \( r > 0 \),

\[
\mu(B(x,2r)) \leq C \mu(B(x,r)),
\]

and that the doubling condition is a key assumption in the classical theory of harmonic analysis. In recent years, many classical results concerning the theory of Calderón-Zygmund operators and function spaces have been proved to be still valid if the Lebesgue measure is substituted by a measure \( \mu \) as in (1.1); see [18–25]. We mention that the analysis on non-homogeneous spaces play an essential role in solving the long-standing open Painlevé’s problem by Tolsa in [21].

To outline the structure of this paper, we first recall some notation and definitions. For a cube \( Q \subset \mathbb{R}^d \), we mean a closed cube whose sides parallel to the coordinate axes, and we denote its side length by \( l(Q) \) and its center by \( x_Q \). Let \( \gamma > 1 \) and \( \beta > \gamma^n \). We say that a cube \( Q \) is an \( (\gamma, \beta) \)-doubling cube if \( \mu(\gamma Q) \leq \beta \mu(Q) \), where \( \gamma Q \) denotes the cube with the same center as \( Q \) and \( l(\gamma Q) = \gamma l(Q) \). For definiteness, if \( \gamma \) and \( \beta \) are not specified, by a doubling cube we mean a \((2,2^d+1)\)-doubling cube. Especially, for any given cube \( Q \), we denote by \( \hat{Q} \) the smallest doubling cube which contains \( Q \) and has the same center as \( Q \).