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Existence of Solution for a Coupled System of Fractional Integro-Differential Equations on an Unbounded Domain

Azizollah Babakhani*

Department of Mathematics, Faculty of Basic Science, Babol University of Technology, Bobol-Iran

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Abstract. We present the existence of solution for a coupled system of fractional integro-differential equations. The differential operator is taken in the Caputo fractional sense. We combine the diagonalization method with Arzela-Ascoli theorem to show a fixed point theorem of Schauder.

Key Words: Fractional derivative/integral, coupled system, Volterra integral equation, diagonalization method.

AMS Subject Classifications: 34LXX, 34GXX

1 Introduction

Fractional differential equations have gained considerable importance due to their various applications in visco-elasticity, electro-analytical chemistry and many physical problems [1–3]. So far there have been several fundamental works on the fractional derivative and fractional differential equations, written by Miller and Ross [4], Podlubny [5] and others in [6–8]. Mathematical aspects of fractional order differential equations have been discussed in details by many authors [9–17].

Consider the Volterra integral equation of the second kind of the form:

$$u(t)=\lambda\int_0^t K(t,s)ds+f(t),$$

where f, K are given functions, λ is a parameter and u is the solution. This equation arises very often in solving various problems of mathematical physics, especially in describing

*Corresponding author. *Email address:* `babakhani@nit.ac.ir` (A. Babakhani)

physical processes after effects [23,24]. Rabha W. Ibrahim and Shaher Momani [25] discussed the upper and lower bounds of solutions for fractional integral equations of the form:

$$u^m(t)=a(t)I^\alpha\{b(t)u(t)\}+f(t), \quad m\geq 1,$$

where $a(t)$, $b(t)$, $f(t)$ are real positive functions in $C([0,T],\mathbf{R})$ and $\alpha\in(0,1)$. Jinhua Wang et al. have investigated the existence and uniqueness of positive solution to nonzero boundary valued problem for a coupled system of nonlinear fractional equation and the reader is referted to [18]. A. Arara et al. [19] have considered a class of boundary valued problems involving Caputo fractional derivative on the half line by using the diagonalization process.

In this paper, we investigate the existence of solution for the coupled system of nonlinear fractional differential equation:

$$^cD^\alpha x(t)=tI^\gamma f(t,y(t))+f(t,y(t)), \quad t\in(0,\infty), \tag{1.1a}$$

$$^cD^\beta y(t)=tI^\eta g(t,x(t))+g(t,x(t)), \quad t\in(0,\infty), \tag{1.1b}$$

$$x(0)=x_0,\ y(0)=y_0,\ x(t) \text{ and } y(t) \text{ are bounded on } [0,\infty), \tag{1.1c}$$

where $1<\alpha,\beta\leq 2$, $^cD^\alpha$ and $^cD^\beta$ are the Caputo fractional derivatives, γ,η are real positive numbers, I^γ and I^η are Riemann-Liouville fractional integral and $f,g:[0,\infty)\times\mathbf{R}\to\mathbf{R}$ are given continuous functions.

2 Basic definitions and preliminaries

We begin in this section to recall some notations, definitions and results for fractional calculus which are used throughout this paper [4,5,7,20].

Let $\mathcal{J}_n=[0,n]$, $L^1(\mathcal{J}_n,\mathbf{R})$ denote the Banach space of functions $x:\mathcal{J}_n\to\mathbf{R}$ that are Lebesgue integrable with the norm

$$\|x\|_{L^1}=\int_0^n|x(t)|dt.$$

Recall that $C(\mathcal{J}_n,\mathbf{R})$ is the Banach space of continuous functions from $\mathcal{J}_n$ into $\mathbf{R}$ endowed with the uniform norm,

$$\|x\|_n=\max\big\{|x(t)|:t\in\mathcal{J}_n\big\},$$

and $C^2=C\times C$ is the Banach space of continuous functions from J_n into $\mathbf{R}$ endowed with the uniform norm

$$\|(x,y)\|_n=\max\big\{\|x\|_n,\ \|y\|_n:(x,y)\in C^2,\ t\in\mathcal{J}_n\big\}.$$

The Arzela-Ascoli theorem and Schauder fixed point theorem are recalled in the following. They play important roles in this article and the reader is referred to [20,21].

Theorem 2.1. (Arzela-Ascoli Theorem). *Let U be a compact metric space and Ω any subset of $C(U)$. Then the following statements are equivalent:*