Some Results Concerning Growth of Polynomials
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Abstract. Let \( P(z) \) be a polynomial of degree \( n \) having no zeros in \(|z| < 1\), then for every real or complex number \( \beta \) with \(|\beta| \leq 1\), and \(|z| = 1\), \( R \geq 1 \), it is proved by Dewan et al. [4] that

\[
|P(Rz) + \beta \left( \frac{R+1}{2} \right)^n P(z)| \leq \frac{1}{2} \left\{ \left( |R^n + \beta \left( \frac{R+1}{2} \right)^n| + |1 + \beta \left( \frac{R+1}{2} \right)^n| \right) \max_{|z|=1} |P(z)| \right. \\
- \left. \left( \left| R^n + \beta \left( \frac{R+1}{2} \right)^n \right| - \left| 1 + \beta \left( \frac{R+1}{2} \right)^n \right| \right) \min_{|z|=1} |P(z)| \right\}
\]

In this paper we generalize the above inequality for polynomials having no zeros in \(|z| < k\), \( k \leq 1 \). Our results generalize certain well-known polynomial inequalities.

Key Words: Polynomial, inequality, maximum modulus, growth of polynomial.

AMS Subject Classifications: 30A10, 30C10, 30E15

1 Introduction and statement of results

It is well known that if \( P(z) \) is a polynomial of degree \( n \), then for \(|z| = 1\) and \( R \geq 1 \)

\[
|P(Rz)| + |Q(Rz)| \leq (R^n + 1) \max_{|z|=1} |P(z)|,
\]

where \( Q(z) = z^n P(1/z) \) (see [6]).

On the other hand, concerning the estimate of \( |P(z)| \) on the disc \(|z| \leq R\), \( R \geq 1 \), we have, as a simple consequence of the principle of maximum modulus (see also [6]), if \( P(z) \) is a polynomial of degree \( n \), then for \( R \geq 1 \)

\[
\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)|.
\]
The result is best possible and the equality holds for polynomials having zeros at the origin.

It was shown by Ankeny and Rivlin [1] that if \( P(z) \) do not vanish in \( |z| < 1 \), then the inequality (1.2) can be replaced by

\[
\max_{|z|=R} |P(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |P(z)|, \quad R \geq 1.
\]  

(1.3)

The inequality (1.3) is sharp and the equality holds for \( P(z) = az^n + \gamma \), where \(|a| = |\gamma|\).

The inequality (1.3) was generalized by Jain [5] who proved that if \( P(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then for \( |\beta| \leq 1, R \geq 1 \) and \(|z|=1\),

\[
\left| P(Rz) + \beta \left( \frac{R+1}{2} \right)^n P(z) \right| 
\leq \frac{1}{2} \left\{ \left| R^n + \beta \left( \frac{R+1}{2} \right)^n \right| + \left| 1 + \beta \left( \frac{R+1}{2} \right)^n \right| \right\} \max_{|z|=1} |P(z)|.
\]  

(1.4)

Aziz and Dawood [3] used

\[
\min_{|z|=1} |P(z)|
\]  

(1.5)

to obtain a refinement of the inequality (1.3) and proved, if \( P(z) \) is a polynomial of degree \( n \) which does not vanish in \( |z| < 1 \), then for \( R \geq 1 \)

\[
\max_{|z|=R} |P(z)| \leq \left( \frac{R^n+1}{2} \right) \max_{|z|=1} |P(z)| - \left( \frac{R^n-1}{2} \right) \min_{|z|=1} |P(z)|.
\]  

(1.6)

The result is best possible and the equality holds for \( P(z) = az^n + \gamma \) with \(|a| = |\gamma|\).

As refinement of the inequality (1.4) and generalization of the inequality (1.6), Dewan and Hans [4] have proved that if \( P(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then for \( |\beta| \leq 1, R \geq 1 \) and \(|z|=1\),

\[
\left| P(Rz) + \beta \left( \frac{R+1}{2} \right)^n P(z) \right| 
\leq \frac{1}{2} \left\{ \left| R^n + \beta \left( \frac{R+1}{2} \right)^n \right| + \left| 1 + \beta \left( \frac{R+1}{2} \right)^n \right| \right\} \max_{|z|=1} |P(z)| 

- \left( \left| R^n + \beta \left( \frac{R+1}{2} \right)^n \right| - \left| 1 + \beta \left( \frac{R+1}{2} \right)^n \right| \right) \min_{|z|=1} |P(z)| \right\}.
\]  

(1.7)

The result is best possible and the equality holds for \( P(z) = az^n + \gamma \) with \(|a| = |\gamma|\).

Whereas if \( P(z) \) has all its zeros in \( |z| \leq 1 \), then for any \( |\beta| \leq 1, R \geq 1 \) and \(|z|=1\),

\[
\min_{|z|=1} |P(Rz) + \beta \left( \frac{R+1}{2} \right)^n P(z) | \geq \left| R^n + \beta \left( \frac{R+1}{2} \right)^n \right| \min_{|z|=1} |P(z)|.
\]  

(1.8)

The result is best possible and the equality holds for \( P(z) = ma \alpha z^n, m > 0 \).

In this paper, we obtain further generalizations of the inequalities (1.7) and (1.8). More precisely, we prove