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THE LOWER DENSITIES OF SYMMETRIC PERFECT SETS

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Abstract. In this paper, we give the exact lower density of Hausdorff measure of a class of symmetric perfect sets.

Key words: *lower density, Hausdorff measure, symmetric perfect set*

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1 Introduction

Let $0 \leq s < \infty$ and ν be a measure on $\mathbf{R}^n$. The upper and lower s-densities of ν at $x \in \mathbf{R}^n$ are defined as

$$\Theta^{*s}(\nu, x) = \limsup_{r \to 0} \frac{\nu(B(x,r))}{(2r)^s},$$

and

$$\Theta^s_*(\nu, x) = \liminf_{r \to 0} \frac{\nu(B(x,r))}{(2r)^s},$$

respectively, where $B(x,r)$ denotes the closed ball with diameter $2r$ and center x.

Symmetric perfect sets are nowhere dense perfect subsets of $[0,1]$ constructed in the following manner.Suppose $I = [0,1]$, let $\{c_k\}_{k \geq 1}$ be a real number sequence satisfying $0 < c_k < \dfrac{1}{2}(k \geq 1)$. For any $k \geq 1$, let

$$D_k = \{(i_1, \cdots, i_k) : \ i_j \in \{1,2\}, \ D = \bigcup_{k \geq 0} D_k,$$

where $D_0 = \emptyset$. If

$$\sigma = (\sigma_1, \cdots, \sigma_k) \in D_k, \quad \tau = (\tau_1, \cdots, \tau_m) \in D_m,$$

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let

$$\sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m).$$

Let $\mathcal{F} = \{I_\sigma : \ \sigma \in D\}$ be the collection of the closed sub-intervals of I satisfying

i) $I_\emptyset = I$;

ii) For any $k \geq 1$ and $\sigma \in D_{k-1}$, $I_{\sigma * i}$ $(i = 1, 2)$ are sub-intervals of I_σ. $I_{\sigma*1}, I_{\sigma*2}$ are arranged from the left to the right, $I_{\sigma*1}$ and I_σ have the same left endpoint, $I_{\sigma*2}$ and I_σ have the same right endpoint.

iii) For any $k \geq 1$ and $\sigma \in D_{k-1}$, $j = 1, 2$, we have

$$\frac{|I_{\sigma * j}|}{|I_\sigma|} = c_k,$$

where $|A|$ denotes the diameter of A.

Let

$$E_k = \bigcup_{\sigma \in D_k} I_\sigma, \qquad E = \bigcap_{k \geq 0} E_k,$$

we call E the symmetric perfect set and call $\mathcal{F}_k = \{I_\sigma : \ \sigma \in D_k\}$ the k-order basic intervals of E. The middle-third Cantor set is a well-known example of the symmetric perfect set.

Let x_k be the length of a k-order basic interval, y_k the length of the gap between any two consecutive sub-intervals $I_{\sigma*1}$ and $I_{\sigma*2}$, where $\sigma \in D_{k-1}$. Assume that

(1) There exists $k_0 \in \mathbf{N}$ such that

$$c_k \leq \frac{1}{3}$$

for all $k > k_0$.

(2) $\lim\limits_{k \to \infty} 2^k x_k^s$ exists and is positive finite.

In [8], we gave a formula to calculate the upper s-density of Hausdorff measure for a class of symmetric perfect sets.

Theorem 1. *Let E be a symmetric perfect set, if (1) and (2) hold, then*

$$\Theta^{*s}(\mu_E, x) = \frac{2}{2^s(2^{\frac{1}{s}} - 1)^s} \quad for \quad \mu_E - \text{a. e.} \quad x \in E,$$

where μ_E is the restriction of the Hausdorff measure $\mathcal{H}^s$ over the set E and s is the Hausdorff dimension of the set E.

This paper gives an analogue for the lower s-density of the Hausdorff measure. Our main result is

Theorem 2. *Let E be the symmetric perfect set, if (2) holds, then*

$$\Theta_*^s(\mu_E, x) = \frac{1}{2^s(2^{\frac{1}{s}} - 1)^s} \quad for \quad \mu_E - \text{a. e.} \quad x \in E.$$