Anal. Theory Appl.
Vol. 28, No. 2 (2012), 180–188
DOI : 10.3969/j.issn.1672-4070.2012.02.008

# ON THE ZEROS OF A CLASS OF POLYNOMIALS AND RELATED ANALYTIC FUNCTIONS

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Received Mar. 24, 2012; Revised Apr. 10, 2012

**Abstract.** In this paper we prove some interesting extensions and generalizations of Enestrom-Kakeya Theorem concerning the location of the zeros of a polynomial in a complex plane. We also obtain some zero-free regions for a class of related analytic functions. Our results not only contain some known results as a special case but also a variety of interesting results can be deduced in a unified way by various choices of the parameters.

**Key words:** *zeros of a polynomial, bounds, analytic functions, moduli of zeros*

**AMS (2010) subject classification:** 30C10, 30C15

## 1 Introduction and Statement of Results

The following well-known result is due to Enestrom and Kakeya[7].

**Theorem A.** *If* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots\cdots + a_1z + a_0$ *is a polynomial of degree n, such that* $a_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0 > 0$, *then* $P(z)$ *has no zeros in* $|z| < 1$.

With the help of Theorem $A$, one gets the following equivalent form of Enestrom-Kakeya Theorem by considering the polynomial $z^nP(1/z)$.

**Theorem B.** *If*

$$P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots\cdots + a_1z + a_0$$

*is a polynomial of degree n, such that*

$$a_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0; \qquad a_0 > 0,$$

*then* $P(z)$ *has no zeros in* $|z| < 1$.

In the literature[1, 4–10], there already exist some extensions and generalizations of Enestrom-Kakeya Theorem. Aziz and Zarger[3] relaxed the hypothesis of Theorem A in several ways and have proved some extensions and generalizations of this result. As a generalization of Enestrom-Kakeya Theorem, they proved:

**Theorem C.** *If* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots\cdots + a_1z + a_0$ *is a polynomial of degree n, such that for some* $k \geq 1$

$$ka_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0 > 0, \tag{1}$$

*then* $P(z)$ *has all its zeros in the disk* $| z + k - 1 | \leq k$.

*Remark* 1. Since the circle $| z + k - 1 | \leq k$ is contained in the circle $| z | \leq 2k - 1$, it follows from Theorem C that all the zeros of $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$, satisfying (I) lie in the circle.

$$| z | \leq 2k - 1. \tag{2}$$

Aziz and Mohammad[2] have studied the zeros of a class of related analytic functions and among other things have obtained.

**Theorem D.** *Let* $f(z) = \sum_{j=0}^{\infty} a_jz^j \neq 0$ *be analytic in* $| z | \leq t$. *If* $| \arg a_j | \leq \alpha \leq \pi/2$, $j = 0, 1, 2, \cdots$ *and for some finite non-negative integer k,*

$$| a_0 | \leq t | a_1 | \leq \cdots \leq t^k | a_k | \geq t^{k+1} | a_{k+1} | \geq \cdots,$$

*then* $f(z)$ *does not vanish in*

$$| z | \leq \frac{t}{\left(2t^k\left|\frac{a_k}{a_0}\right| - 1\right)\cos\alpha + \sin\alpha + \frac{2\sin\alpha}{| a_0 |} \mid \sum_{j=0}^{\infty} t^j \mid a_j \mid}.$$

The aim of this paper is to present some more extensions and generalizations of Enestrom-Kakeya Theorem. We also study the zeros of a class of related analytic functions. We start by presenting the following interesting generalization of Theorem C.

**Theorem 1.** *If* $P(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots\cdots + a_1z + a_0$ *is a polynomial of degree n. If for some real number* $\rho \geq 0$, *such that*

$$\rho + a_n \geq a_{n-1} \geq \cdots\cdots \geq a_1 \geq a_0 > 0, \tag{3}$$

*then* $P(z)$ *has all its zeros in*

$$| z + \frac{\rho}{a_n} | \leq 1 + \frac{\rho}{a_n}. \tag{4}$$

*Remark* 2. Theorem C is a special case of Theorem 1 for the choice of $\rho = (k-1)a_n$, where $k \geq 1$. Applying Theorem 1 to polynomial $P(tz)$ we obtain the following result :