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CONVERGENCE OF DERIVATIVES OF GENERALIZED BERNSTEIN OPERATORS

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Abstract. In the present paper, we obtain estimations of convergence rate derivatives of the q-Bernstein polynomials $B_n(f,q_n;x)$ approximating to $f^{'}(x)$ as $n\to\infty$, which is a generalization of that relating the classical case $q_n=1$. On the other hand, we study the convergence properties of derivatives of the limit q-Bernstein operators $B_\infty(f,q;x)$ as $q\to 1^-$.

Key words: *limit q-Bernstein operators, derivative of q-Bernstein polynomial, convergence, rate*

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1 Introduction

For an integer $r\geq 0$, let $C^r[0,1]$ be the class of all functions $f(x)$ which have the r-th continuous derivatives on $[0,1]$, where $C^0[0,1]=C[0,1]$ is the usual class of all continuous functions on $[0,1]$ with the supremum norm$\|.\|$.

Let $q>0$. For any $n=0,1,2,\cdots$, the q-integer $[n]_q$ is defined as

$$[n]_q=1+q+\cdots+q^{n-1},\qquad n=1,2,\cdots,\quad [0]_q=0$$

and the q-factorial $[n]_q!$ as

$$[n]_q!=[1]_q[2]_q\cdots[n]_q,\qquad n=1,2,\cdots,\quad [0]_q!=1;$$

For the integers n,k, $n\geq k\geq 0$, the q-binomial, or the Gaussian coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{[n]_q!}{[k]_q![n-k]_q!}.$$

In 1997, Phillips proposed the generalized Bernstein polynomials (see [7]), or the q-Bernstein polynomials $f(x)\in C[0,1]$,

$$B_n(f,q;x)=\sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)p_{n,k}(q;x),\qquad n=1,2,\ldots, \tag{1.1}$$

where

$$p_{n,k}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x), \qquad k = 0,1,2,\cdots n. \tag{1.2}$$

(From here on, an empty product denotes 1, and an empty sum denotes 0).

When $q = 1$, $B_n(f,q;x)$ reduce to the classical Berntein polynomials

$$B_n(f,1;x) = B_n(f,x) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k (1-x)^{n-k}, \qquad n = 1,2\ldots,$$

where $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_1$ is the classical binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n!}{k!(n-k)!}.$$

For $f \in C[0,1], t > 0$, we define the modulus of continuity $\omega(f,t)$ and second modulus of smoothness $\omega_2(f,t)$ as follows:

$$\begin{aligned} \omega(f,t) &= \sup_{|x-y|\le t, x,y\in[0,1]} |f(x)-f(y)|, \\ \omega_2(f,t) &= \sup_{0<h\le t} \sup_{x\in[0,1-2h)} |f(x+2h)-2f(x+h)+f(x)|. \end{aligned}$$

In the sequel, $C, C_1, C_{2,}, \cdots$ denote positive constants (difference at different occurrences).

Recently, it is found that q-Bernstein polynomials possess many remarkable properties (see[3,4,6-10,13]), which make them an area of intensive research.

In [3], Il'inskii and Ostrovska proved that for $f(x) \in C[0,1]$, $B_n(f,q;x)$ converge to $B_\infty(f,q;x)$ as $n \to \infty$ uniformly with respect to $x \in [0,1]$, and $q \in [\alpha,1]$, $0 < \alpha < 1$, where

$$B_\infty(f,1;x) = f(x), \qquad x \in [0,1],$$

and for $q \in (0,1)$,

$$B_\infty(f,q;x) = \begin{cases} \sum\limits_{k=0}^{\infty} f(1-q^k) p_{\infty,k}(q;x), & 0 \le x < 1, \\ f(1), & x = 1, \end{cases} \tag{1.3}$$

which we call the limit q-Bernstein operators (see [5]), where

$$p_{\infty,k}(q;x) = \frac{x^k}{(1-q)^k [k]_q!} \prod_{s=0}^{\infty}(1-q^s x), \qquad k = 0,1,\cdots. \tag{1.4}$$

The limit q-Bernstein operator is a positive shape-preserving linear operator approximating continuous functions on [0,1] as $q \to 1^-$. A large number of results relating to various properties of these operators have been obtained (see [3,5,12]).

In [11], for the derivative of classical Bernstein polynomials, L.Xie obtained the following result.