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WEAK TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED NON-HOMOGENEOUS MORREY SPACES

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Abstract. We obtain weak type $(1,q)$ inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces. The proofs use some properties of maximal operators. Our results are closely related to the strong type inequalities in [13, 14, 15].

Key words: *weak type inequalitiy fractional integral operator, (generalized) non-homogeneous Morrey psace*

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1 Introduction

The work of Nazarov et al.[10], Tolsa[17], and Verdera [18] reveal some important ideas of the spaces of non-homogeneous type. By a non-homogeneous space we mean a (metric) measure space−here we consider only $\mathbf{R}^d$ equipped with a Borel measure μ satisfying the growth condition of order n with $0 < n \leq d$, that is there exists a constant $C > 0$ such that

$$\mu(B(a,r)) \leq C r^n \tag{1}$$

for every ball $B(a,r)$ centered at $a \in \mathbf{R}^d$ with radius $r > 0$. The growth condition replaces the *doubling condition*:

$$\mu(B(a,2r)) \leq C\mu(B(a,r))$$

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which plays an important role in the space of homogeneous type.

In the setting of non-homogeneous spaces described above, we define the fractional integral operator I_α $(0 < \alpha < n \le d)$ by the formula

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}\mu(y)$$

for suitable functions f on $\mathbf{R}^d$. Note that if $n = d$ and μ is the usual Lebesgue measure on $\mathbf{R}^d$, then I_α is the classical fractional integral operator introduced by Hardy and Littlewood[5,6] and Sobolev[16]. The classical fractional integral operator I_α is known to be bounded from the Lebesgue space $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$ where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ for $1 < p < \frac{d}{\alpha}$. This result has been extended in many ways-see for examples [4, 8, 11] and the references therein.

For $p = 1$, we have a weak type inequality for I_α and on non-homogeneous Lebesgue spaces such an inequality has been studied, among others, by García-Cuerva, Gatto, and Martell in [2, 3]. One of their results is the following theorem. (Here and after, we denote by C a positive constant which may be different from line to line.)

Theorem 1.1[2,3]**.** $\frac{1}{q} = 1 - \frac{\alpha}{n}$, *then for any function* $f \in L^1(\mu)$ *we have*

$$\mu\{x \in \mathbf{R}^d : |I_\alpha f(x)| > \gamma\} \le C \left(\frac{\|f\|_{L^1(\mu)}}{\gamma} \right)^q, \qquad \gamma > 0.$$

The proof of Theorem 1.1 uses the weak type inequality for the maximal operator

$$Mf(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)| \, \mathrm{d}\mu(y).$$

In this paper, we shall prove the weak type inequality for I_α on generalized non-homogeneous Morrey spaces (which we shall define later). The proof will employ the following inequality for the maximal operator M.

Theorem 1.2[3,12]**.** *For any positive weight* w *on* $\mathbf{R}^d$ *and any function* $f \in L^1_{\mathrm{loc}}(\mu)$*, we have*

$$\int_{\{x \in \mathbf{R}^d : Mf(x) > \gamma\}} w(x) \, \mathrm{d}\mu(x) \le \frac{C}{\gamma} \int_{\mathbf{R}^d} |f(x)| Mw(x) \, \mathrm{d}\mu(x), \qquad \gamma > 0.$$

Our main results are presented as Theorems 2.2 and 2.3 in the next section. Related results can be found in [13, 14, 15].

2 Main Results

For $1 \le p < \infty$ and a suitable function $\phi : (0,\infty) \to (0,\infty)$, we define the generalized non-homogeneous Morrey space $\mathcal{M}^{p,\phi}(\mu) = \mathcal{M}^{p,\phi}(\mathbf{R}^d,\mu)$ to be that of all functions $f \in L^p_{\mathrm{loc}}(\mu)$ for