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A SIMPLE PROOF OF THE CHAOTICITY OF SHIFT MAP UNDER A NEW DEFINITION OF CHAOS

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Abstract. Recently, Du has given a new strong definition of chaos by using the shift map. In this paper, we give a proof of the main theorem by constructing a dense uncountable invariant subset of the symbol space Σ_2 containing transitive points in a simpler way with the help of a different metric. We also provide two examples, which support this new definition.

Key words: *symbol space, shift map, δ-scrambled set, chaos, transitive points*

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1 Introduction

Nowadays the chaoticity of a dynamical system becomes a more demanding and challenging topic for both mathematicians and physicists. Li and Yorke[8] are the first people that connect the term ‘chaos’ with a map. There are various types of chaotic maps, namely, tent map [6, 10], quadratic map [2, 4, 6, 10] etc. If for a system the distance between the nearby points increases and the distance between the faraway points decreases with time, the system is said to be chaotic. According to Devaney[6], a map $f : V \to V$ is said to be chaotic if the following three properties hold:

(i) f has sensitive dependence on initial conditions;

(ii) f is topologically transitive; and

(iii) periodic points are dense in V.

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But in [3], it is shown that topological transitivity and dense periodic orbits together imply sensitive dependence on initial conditions. So the condition (i) of Devaney's definition is redundant. It is also known that in an interval (not necessarily finite) a continuous, topological transitive map is chaotic in the sense of Devaney[6,11]. Akin[1] introduced a linkage between the sensitivity and Li-Yorke[8] version of chaos.

Let $\Sigma_2 = \{\alpha : \alpha = (\alpha_0\alpha_1\cdots), \alpha_i = 0 \text{ or } 1\}$ be the symbol space containing two symbols 0 and 1. For any two points $s = (s_0s_1\cdots)$ and $t = (t_0t_1\cdots)$ of Σ_2, we define the distance between s and t by the metric

$$d(s,t) = \sum_{k=0}^{\infty} \frac{\delta(s_k,t_k)}{3^{k+1}},$$

where $\delta(s_k,t_k) = 0$, for $s_k = t_k$ and $\delta(s_k,t_k) = 1$, for $s_k \neq t_k$. It can be easily shown that Σ_2 is a metric space with the metric $d(s,t)$. We noted that the maximum distance between two points of Σ_2 with the metric is $\frac{1}{2}$, because

$$d(s,t) \leq \frac{1}{3} + \frac{1}{3^2} + \cdots = \frac{1}{2}.$$

The shift map [4, 6, 7, 9, 10] is often used to model the chaoticity of a dynamical system. Shift map has some remarkable and interesting properties, such as, it is topologically transitive, has dense periodic points and sensitive dependence on initial conditions. The present authors have extend the idea of the shift map to the generalized shift map in [5]. Recently. It is shown by Du[7] that the sensitive dependence on initial conditions tells only a part about the system to be chaotic and so he used another interesting property of the shift map called the extreme sensitive dependence on initial conditions to show a system to be more chaotic than the previous one.

In this paper, we use a metric different from the that used in [7] and with the help of this metric and a different but simple construction of the dense uncountable invariant subsets of Σ_2 containing transitive points, we are able to give a simpler proof of the chaoticity of the shift map. Further, we give two new examples, which illustrate the fact that (i) all topologically transitive maps or Li-Yorke sensitive maps are not chaotic and (ii) chaotic maps are not necessarily topologically transitive. We now state this new definition of chaos, which was given by Du.

2 A Strong Definition of Chaos[7]

Let (X,ρ) be an infinite compact metric space with the metric ρ and let f be a continuous map from X into itself. We say that f is chaotic if there exists a positive number δ such that for any point x and any non-empty open set V (not necessarily an open neighborhood of x) in X