Anal. Theory Appl.
Vol. 27, No. 2 (2011), 187–200
DOI10.1007/s10496-011-0187-3

ON APPROXIMATION OF SMOOTH FUNCTIONS FROM NULL SPACES OF OPTIMAL LINEAR DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

Vesselin Vatchev

(*University of Texas at Brownsville, U. S.)*

Received Nov. 28, 2010; Revised May 7, 2011

Abstract. For a real valued function f defined on a finite interval I we consider the problem of approximating f from null spaces of differential operators of the form $L_n(\psi) = \sum_{k=0}^{n} a_k\psi^{(k)}$, where the constant coefficients $a_k \in \mathbf{R}$ may be adapted to f.

We prove that for each $f \in C^{(n)}(I)$, there is a selection of coefficients $\{a_1, \cdots, a_n\}$ and a corresponding linear combination

$$S_n(f,t) = \sum_{k=1}^{n} b_k e^{\lambda_k t}$$

of functions $\psi_k(t) = e^{\lambda_k t}$ in the nullity of L which satisfies the following Jackson's type inequality:

$$\|f^{(m)} - S_n^{(m)}(f,t)\|_\infty \le \frac{|I|^{1/q} e^{|\lambda_n||I|}}{|a_n| 2^{n-m-1/p} |\lambda_n|^{n-m-1}} \|L_n(f)\|_p,$$

where $|\lambda_n| = \max_k |\lambda_k|$, $0 \le m \le n-1$, $p, q \ge 1$, and $\frac{1}{p} + \frac{1}{q} = 1$.

For the particular operator $M_n(f) = f + 1/(2n)!f^{(2n)}$ the rate of approximation by the eigenvalues of M_n for non-periodic analytic functions on intervals of restricted length is established to be exponential. Applications in algorithms and numerical examples are discussed.

Key words: *approximation of analytic function, differential operator, fundamental set of solutions*

AMS (2010) subject classification: 41A80, 41A58

Supported in part by grant HRD-0734800.

1 Introduction

The problem of approximating a real valued function f on a finite interval I by linear combinations from subsets of the set $E = \{e^{\lambda t} | \lambda - \text{complex number}\}$ is studied extensively. The most classical approach is the Fourier series expansion when one uses expanding subspaces $S_N = \text{span}\{1, e^{\pm it}, \dots, e^{\pm iNt}\}$ and the linear combination is the solution of the extremal problem

$$ET_N(f) = \min_{c_k \in \mathbf{R}} \|f(t) - \sum_{k=-N}^{N} c_k e^{ikt}\|_2(I),$$

where

$$\|f\|_p(I) = \left(\int_I |f(t)|^p \mathrm{d}t\right)^{1/p}, \qquad 0 < p < \infty$$

and $\|f\|_\infty = \inf_{t \in I} |f(t)|$ are the norms of f in the spaces $L_p(I), 0 < p \leq \infty$. The exponents ik lie on the imaginary axis and are predetermined for any function f. The "goodness" of approximation is measured by the rate at which $ET_N(f)$ approaches 0 as $N \to \infty$. To characterise the classes of functions with the same rate of approximation for smooth functions it is important to establish Jackson's type estimates of the form $ET_N(f) \leq \dfrac{\|f^{(m)}\|_2}{N^\alpha}$, in L_2 for example. The maximum α determines the rate of approximation, for more details see [6]. One way to generalize the classical Fourier series is to replace S_N by the set of eigenfunctions of predetermined differential operators, Sturm-Liouville and self-adjoint operators, and etc., for details see [1].

The goal of the present paper is to study approximation by finite linear combinations of elements of E with exponents $\lambda_k, k = 1, \dots, n$ adapted to f. For a given function f on a finite interval I we determine λ's as the characteristic roots of the linear differential operator $L_n(\psi, t) = \sum_{k=0}^{n} a_k \psi^{(k)}(t), a_k \in \mathbf{R}$, that minimizes $\|L_n(f, t)\|_2(I)$. For a particular operator L_n the space of solutions to the equation $L_n(\psi, t) = 0$ is called null space of L_n or fundamental set of solutions. That space is spanned by n functions, $\psi_k(t) = e^{\lambda_k t}, k = 1, \dots, n$, where λ_k are the roots of the corresponding characteristic polynomial $P_n(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$. Once the optimal operator L_n is determined we approximate f in $L_p, p > 0$ by linear combinations from the finite dimensional null space of L_n. Fourier and polynomial approximations could be considered in that setting with predetermined operators. For the particular sequence of operators $L_n(f) = f^{(n)}$ the fundamental set of solutions are the power functions $\psi_k(t) = t^k, k = 0, \dots, n-1$. For the operators $S_N = \prod_{k=-N}^{N} (D + ikId)$, where $i^2 = -1, D^k f = f^{(k)}$ and $Id(f) = f$ is the identity operator the fundamental set of solutions is the set of the Fourier modes $e^{ikt}, k = -N, \cdots, N$.

It is well known, see [6], that if the function f is non-periodic on I the rate of convergence of the Fourier series is $1/n$. In Section 2 we establish exponential decay of the rate of approximation