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ON CERTAIN PROPERTIES OF THE COMBINATIONS OF SZÁSZ-DURRMEYER OPERATORS

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Abstract. In the present paper we study properties of Szász-Durrmeyer operators. These operators are introduced in [5] and generalize the integral operators proposed by S.M.Mazhar and V.Totik in [12]. We also generalize some results obtained by M. Heilmann[6] and D.-X. Zhou[16].

Key words: *modified Szász-Mirakyan operator, Baskakov-Durremeyer operator*

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1 Introduction

In this paper we study Szász-Durrmeyer operators defined on functions $f \in L_p$ in the following form:

$$M_{n,\nu}(f,x) = \int_0^\infty n \sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k+\nu}(t) f(t) \mathrm{d}t,$$

where $x,\ n \in [0,\infty)$, $k \in \mathbb{N}$, $\nu \in (-1,\infty)$ and $p_{n,l}(t) = e^{-nx}\dfrac{(nx)^l}{\Gamma(l+1)}$ for $l \in [0,\infty)$. The term

$$K_{n,\nu}(t,x) = n \sum_{k=0}^{\infty} p_{n,k}(x) p_{n,k+\nu}(t)$$

is called the kernel of Szász-Durrmeyer operator. This family of operators was introduced by A. Ciupa and I.Gavrea[15] and was also independently proposed by E. Wachnicki[5]. Some typical results could be found in papers [5, 13, 14, 7]. In the first section we placed the results which are useful in the proof of the further theorems, some of them generalize the properties which are known for the particular case (for $\nu = 0$, $n \in \mathbf{N}$) but others couldn't even be formulated for the previously considered families of operators. As the main results in the second section we present certain theorems for combinations of considered operators. Similar results (theorems 3,4 and 5) were obtained by M. Heilmann[6] and later by D. Zhou[16] who used the same method for particular case of Szász-Durrmeyer operators (for $\nu = 0$, $n \in \mathbf{N}$). In our proofs we use the same ideas, but they are slightly simplified compared to the mentioned special case [16] thanks to Theorem 1 and Lemma 4.

2 Auxiliary Results

By simple induction with respect to r we obtain the following

Lemma 1. *Let $r \in \mathbf{N}$, $x, n, k \in [0, \infty)$, then*

$$p_{n,k}^{(r)}(x) = n^r \sum_{j=0}^{r} (-1)^{r-j} \begin{pmatrix} r \\ j \end{pmatrix} p_{n,k-j}(x), \quad r \le k, \tag{1}$$

$$x p_{n,k}^{(r+1)}(x) = (k - nx) p_{n,k}^{(r)}(x) - r \left(p_{n,k}^{(r)}(x) + n p_{n,k}^{(r-1)}(x) \right), \tag{2}$$

$$p_{n,k}^{(r+1)}(x) = n[p_{n,k-1}^{(r)}(x) - p_{n,k}^{(r)}(x)]. \tag{3}$$

Moreover, $p_{n,k}^{(r)}(x)$ are in the form

$$x^r p_{n,k}^{(r)}(x) = p_{n,k}(x) \sum_{i=0}^{r} \sum_{j=0}^{[\frac{i}{2}]} b_{r,i,j} (k - nx)^{r-i} (nx)^j, \tag{4}$$

where $i \in \{0, 1, ..., r\}$, $j \in \{0, 1, ..., [\frac{i}{2}]\}$ and $b_{r,i,j}$ are coefficients independent of x, n, k.

Using properties of the Gamma-Euler function, we get

$$\int_0^\infty t^s p_{n,k+\nu}(t) \mathrm{d}t = n^{-(s+1)} \frac{\Gamma(k+\nu+s+1)}{\Gamma(k+\nu+1)}, \qquad \text{for } k \in \mathbf{N},\ s, n \in \mathbf{N}^* \tag{5}$$

and

$$n \int_0^\infty p_{n,k+\nu}(t) \mathrm{d}t = 1. \tag{6}$$

Let us note that

$$\frac{\Gamma(k+\nu+s+1)}{\Gamma(k+\nu+1)} = (k+\nu+1) \cdot \ldots \cdot (k+\nu+s) = \sum_{i=0}^{s} k^i B_{s,i}^{\nu}, \tag{7}$$

where $B_{s,i}^{\nu}$ are coefficients of the polynomial independent of i, s and ν, moreover $B_{s,s}^{\nu} = 1$.

Taking $B_{s,i}^{\nu} = 0$ for $i < 0$ or $i > s$ we obtain

$$B_{s+1,i}^{\nu} = (\nu+1) B_{s,i}^{\nu+1} + B_{s,i-1}^{\nu+1}. \tag{8}$$