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On the Degree of Approximation of Continuous Functions by Means of Fourier Series in the Hölder Metric

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Abstract. In this paper we prove two theorems on the degree of approximation of continuous functions by matrix means related to partial sums of a Fourier series in the Hölder metric. These theorems can be taken as counterparts of those previously obtained by T. Singh [3].

Key Words: Matrix transformation, degree of approximation, Fourier series, Hölder metric.

AMS Subject Classifications: 42A24, 42B05, 42B08

1 Introduction and the aim of the paper

The space of all 2π- periodic continuous functions f on $[0,2\pi]$ with Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

is denoted by $C_{2\pi}$.

The space H_ω is defined by

$$H_\omega = \{f \in C_{2\pi} : |f(x) - f(y)| \le K\omega(|x-y|)\},$$

while the norm $\|\cdot\|_{\omega^*}$ is defined by

$$\|f\|_{\omega^*} = \|f\|_C + \sup_{x,y} \Delta^{\omega^*} f(x,y),$$

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where

$$\|f\|_C = \sup_{0\le x\le 2\pi} |f(x)|,$$
$$\Delta^{\omega^*} f(x,y) = \frac{|f(x)-f(y)|}{\omega^*(|x-y|)}, \qquad x \neq y,$$

and $\Delta^0 f(x,y)=0$.

The functions $\omega(t)$ and $\omega^*(t)$ are assumed to be increasing functions of t. If $\omega(|x-y|) \le K_1|x-y|^\alpha$ and $\omega^*(|x-y|) \le K_2|x-y|^\beta$, $0 \le \beta < \alpha \le 1$, where K_1, K_2 are positive constants, then the space

$$H_\alpha = \{f \in C_{2\pi} : |f(x)-f(y)| \le K|x-y|^\alpha,\ 0<\alpha\le 1\}$$

is a Banach space [8] and the metric induced by the norm $\|\cdot\|_\alpha$ on H_α is said to be a Hölder metric.

Let $S_n(f;x)$ denote the n-th partial sum of Fourier series of the function f. Let $A := (a_{n,k})$ $(k,n=0,1,\cdots)$ be a lower triangular infinite matrix of real numbers and let the A-transform of $\{S_n(f;x)\}$ be given by

$$T_{n,A}(f) := T_{n,A}(f;x) := \sum_{k=0}^{n} a_{n,k} S_k(f;x), \quad (n=0,1,\cdots).$$

The following notations will be used later:

$$\phi_x(t) := f(x+t)+f(x-t)-2f(x),$$
$$D_{n,A}(f) := T_{n,A}(f) - f.$$

Seemingly, was Chandra [1] who for the first time extended Prössdorf's [8] results to find the degree of approximation of a continuous function using the Nörlund transform. Later on, Mohapatra and Chandra [2] obtained a number of interesting results on the degree of approximation in the Hölder metric using matrix transforms, which generalize all the previous results based on Cesàro and Nörlund transforms. Their result can be read as follows:

Theorem 1.1. *Let $A := (a_{n,k})$ $(k,n=0,1,\cdots)$ be a lower triangular infinite matrix such that*

$$a_{n,k} \ge 0, \quad n,k=0,1,2,\cdots, \quad \text{and} \quad \sum_{k=0}^{n} a_{n,k} = 1, \tag{1.1a}$$
$$a_{n,k} \le a_{n,k+1}, \quad k=0,1,2,\cdots,n-1; \quad n=0,1,2,\cdots. \tag{1.1b}$$

Then for $f \in H_\alpha$, $0 \le \beta < \alpha \le 1$

$$\|D_{n,A}(f)\|_\beta = \begin{cases} \mathcal{O}(n^{\beta-\alpha}) + \mathcal{O}(a_{n,n} n^{\beta-\alpha+1}), & 0<\alpha<1, \\ \mathcal{O}(n^{\beta-1}) + \mathcal{O}(a_{n,n} n^{\beta} (\log n)^{1-\beta}), & \alpha = 1. \end{cases}$$